Large PHiZZy Donuts & Cola Modelling Surfaces with Modular Origami

by

Lee Jian Le, Lim Jia Wei, Lu Yongquan

Mentor: Gwee Hwee Ngee

Hwa Chong Institution (High School)

1 Introduction

Modular origami is the art of assembling multiple similarly folded units to form a single geometric form. Motivated by well-known origami artist and mathematician Thomas Hull's research, we investigated the topological possibilities and geometric properties of assemblies of an origami unit known as the PHiZZ unit. Due to its ability to form various types of discrete Gaussian curvature, we discover that it is possible to assemble PHiZZ units to form several well known 2-manifolds.

2 Curvature and the PHiZZ Unit

PHiZZ actually stands for 'Pentagon Hexagon Zig-Zag', referring to its original application in creating buckyballs like the truncated icosahedron in Fig 1. Its folding and assembly instructions can be found at [1].

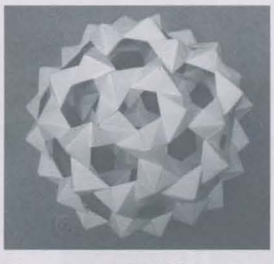


Figure 1. A PHiZZ assembly

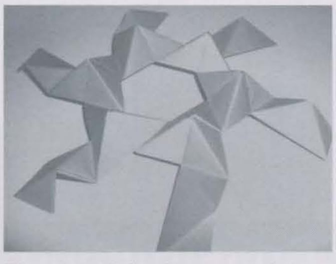


Figure 2. A flat hexagonal ring

From the trivalency of the assembled vertices, we can infer the rings of hexagons lie flat and tile the plane (as demonstrated by the assembly in Fig 2). However, what about rings of other polygons - how do they curve?

By actually assembling these units, we discovered that the rings are actually capable of emulating different types of Gaussian curvature (Fig 3). Gaussian curvature of a point is defined as the product of the maximum and minimum curvatures of all geodesics passing through that point. Hence we observed that pentagons demonstrate positive Gaussian curvature by being locally ellipsoidal (informally, looking like the surface of a sphere). On the other hand, heptagons and larger rings demonstrate negative Gaussian curvature (where the principal curvatures are of opposite polarity). They are hence locally hyperboloidal and can be informally described to be akin to the surface of a horse saddle.

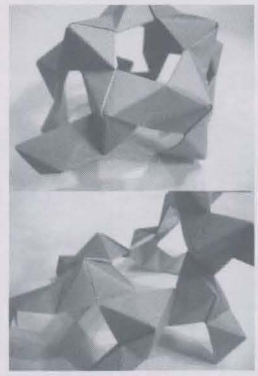


Figure 3. Pentagonal and octagonal rings express curvature

3 Professor Tom Hull's Modular Tori

Analogous to how constructing adjacent pentagonal rings eventually leads to a dodecahedron (smaller rings exert too much stress on the joints to be stable), it is possible to construct heptagonal (or larger n-gonal) tessellations that model the hyperbolic plane. Here we have decided to explore Professor Hull's original research, - how combining polygons can lead to modular tori.

Professor Hull's detailed explanation can be found at [2]. Here we briefly introduce several models and their corresponding designs. In Fig 4, it is shown what we call the "octagonal design", because of its use of octagons in the inner rim to induce negative curvature. The symmetry of this design, invented by Sarah Marie Belcastro, can be more easily seen in Fig 4, where the polygonal arrangement is inscribed in the torus's fundamental polygon, a rectangle with opposite sides identified. This is tiled four times in total to 'close up' the torus.

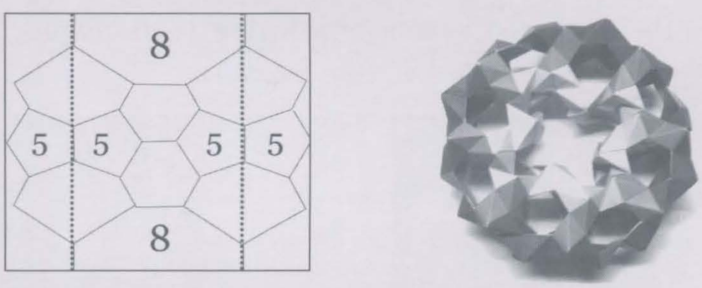


Figure 4. Polygonal design and PHiZZ fourfold construction

4 Constructing a Genus 2 Torus

After seeing how the curvature of these units can model a torus, the natural question to ask is, what other surfaces are constructible? With that in mind, we proceeded to construct a genus 2 torus - informally, a surface with two 'holes'.

By viewing the genus 2 torus as the connected sum of two tori, we adapted portions of the previous octagonal design. Hence, the main question that remained was how the 'middle linkage' expressed curvature and how would we come up with a suitable polygonalisation of the surface.

The first step is to naturally invoke Euler's formula,

$$V - E + F = \chi,$$

where V stands for the number of vertices, E the number of edges, F the number of faces, and χ is the Euler characteristic of the surface. For orientable surfaces,

$$\chi = 2 - 2g,$$

indicating that for the genus 2 torus, we have

$$V - E + F = -2.$$

We note that PHiZZ structures have uniform vertex valency of degree 3, giving us the relationship $3V = 2E$. Also, if we let F_i denote the number of polygons with i sides, we get the following two equations:

$$\sum F_i = F \text{ and } \sum iF_i = 2E.$$

Substitution and algebraic manipulation then gives us

$$\sum \frac{6-i}{6} F_i = \chi.$$

By restricting our choice of polygons to pentagons, hexagons and octagons in our desired genus 2 surface, we obtain

$$F_5 + 12 = 2F_8.$$

With further consideration of the structure of the surface, we derive our polygonalisation (Fig 5, note the similarities with the octagonal genus 1 torus design) and construct the surface successfully.

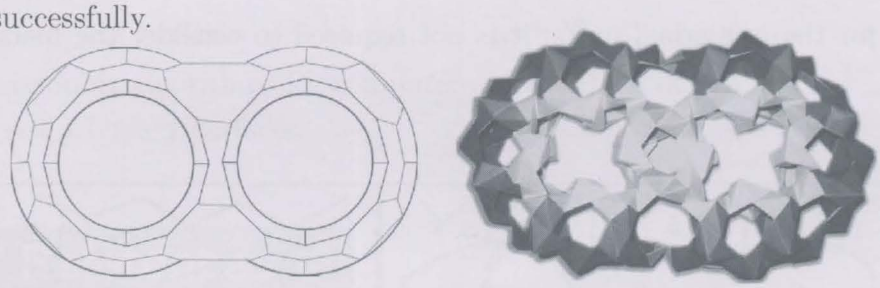


Figure 5. Polygonal arrangement and construction of the genus 2 torus

5 Constructing a Klein Bottle

We realized that via this process of attaching handles, all canonical orientable surfaces can be constructed, and hence turned our attention towards non-orientable surfaces, more specifically the Klein bottle (Fig 6).

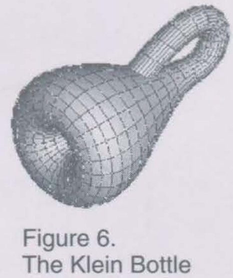


Figure 6.
The Klein Bottle

By adopting the classical bottle immersion of the Klein bottle in 3-space, we note that a nexus, or region of intersection should necessarily occur where the neck enters the body. This should ideally take place from the holes inherent in PHiZZ structures.

Another observation is how the non-orientable nature of the Klein bottle (informally, how it only has one side like the Mobius Strip) conflicts with the orientable nature of PHiZZ surfaces where the vertex pyramids distinguish the two sides. We can resolve this via changing the creases of several units such that they form a seam for units facing one way to join units facing another (Fig 7).

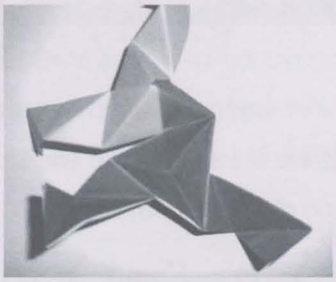


Figure 7. Reversed pyramid direction

On this occasion, we have adopted tertiary units to increase efficiency - designing blocks of polygons that can be reused instead of individual polygonal arrangements for separate areas. There are primarily two regions that we focus on, indicated below in Fig 8: These arrangements were made after applying our above consequence of trivalent polygonalisations,

$$\sum \frac{6-i}{6} F_i = \chi,$$

in this case to obtain

$$F_5 = 2F_8.$$

By ensuring this applies for the individual units, it is not required to consider the macro structure of the surface.

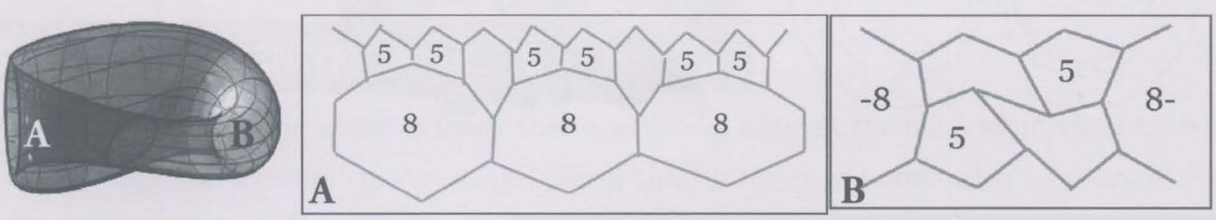
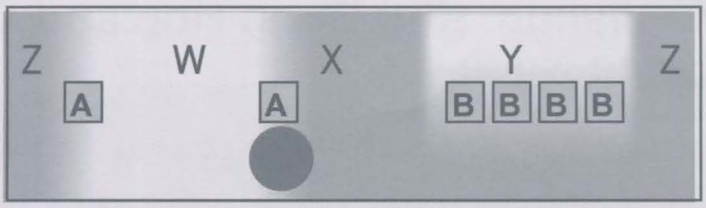


Figure 8. Polygonal arrangements correspond to the regions in the Klein bottle on the right.

Note how the first arrangement (called A) is essentially the body curving in to form the neck, and the second arrangement (dubbed B) the twisting of the elongated neck. We can arrange units of A and B on the curvature plan below, Fig 9, arranged on the fundamental polygon of the Klein bottle, where colour gradients correspond to changes in curvature on the surface.



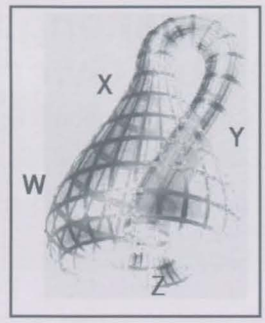


Figure 9. Tertiary units A & B in perspective of the curvature plan of the Klein bottle (dark colours indicate positive curvature and light negative, the circle the nexus). A labeled picture is provided to illustrate how regions correspond.

With this diagram we can then derive our polygonalisation easily, adding several rows of hexagons to increase the 'bulk' of the surface. This Klein bottle requires 153 units and its construction is Fig 10.

6 Conclusion

Successful constructions of the genus 2 torus and the Klein bottle have been achieved by applying topological and geometrical concepts as well as the properties of the PHiZZ units.

We plan to investigate further into proving all canonical topological surfaces can be modeled with PHiZZ units and developing algorithms for computer-aided PHiZZ design for given arbitrary surfaces.

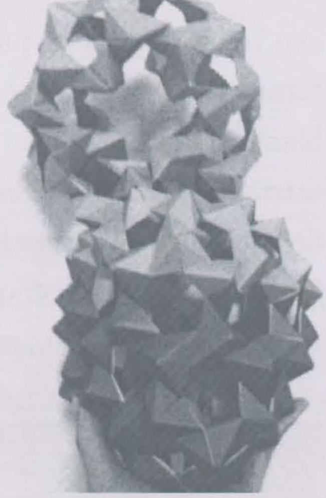


Figure 10. Our Klein Bottle Model

7 References

1 Hull, Thomas. "Origami Math." http://www.merrimack.edu/ thull/phzig/phzig.html
2 Hull, Thomas. "Tom's Combinatorial Geometry Class."
http://www.merrimack.edu/ thull/combgeom/tori/torusnotes.html

Editor's Note

This paper is based on the project submitted by the authors for the Singapore Mathematics Project Festival 2007. The team won the Foo Kean Pew Memorial Prize (Senior Section).