Graphs and Their Applications (10)

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29 System of Distinct Representatives

In Section 27 of our previous article [5], we discussed the following celebrated theorem, namely, **Hall's Theorem** on **matchings** in bipartite graphs, and its applications.

Theorem 27.1 Let G be a bipartite graph with bipartition (X, Y). Then G contains a complete matching from X to Y if and only if $|A| \leq |N(A)|$ holds in G for every subset A of X.

In this section, we shall introduce the notion of system of distinct representatives for a family of finite sets, and prove a classic result in the next section about the existence of such a family by applying Theorem 27.1.

We begin with the following example.

Example 29.1 *In the mathematics department of a university, there are five staff committees with their executives (excluding the Head of Department) elected as shown below:*

$$\begin{array}{lcl} \textit{Colloquium } (C) & : & \{a,b\}, \\ \textit{Library } (L) & : & \{b,c,d\}, \\ \textit{Research } (R) & : & \{a,b\}, \\ \textit{Sports } (S) & : & \{d,e\}, \\ \textit{Teaching } (T) & : & \{b,e\}. \end{array}$$

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The Head would like to call a meeting where each committee is represented by one executive and different committees must be represented by distinct representatives. Can this be done?

By trial and error, it is not difficult to see that it can be done. Indeed, one possible solution is shown below (where '$-$' indicates 'representing'):

$$a - C, c - L, b - R, d - S \text{ and } e - T.$$

Note that C and R have the same set of executives.

The above example provides an instance for the following important notion in Combinatorics.

> Let U be a non-empty set, and $S_1, S_2, \cdots, S_m$ be non-empty finite subsets (*but not necessarily distinct*) of U. A **system of distinct representatives (SDR)** for the family $(S_1, S_2, \cdots, S_m)$ is a sequence of m elements $(a_1, a_2, \cdots, a_m)$ such that $a_i \in S_i$ for each $i = 1, 2, \cdots, m$, and $a_i \neq a_j$ whenever $i \neq j$.

Thus, in Example 29.1, we see that the sequence (a, c, b, d, e) is an SDR for the family (C, L, R, S, T). Note that it is allowable that $S_i = S_j$ for some distinct i and j; for instance, we have $C = \{a, b\} = R$ in Example 29.1.

Example 29.2 *Let U be the set of natural numbers. Consider the family of subsets of U in each of the following cases:*

(i) $S_1 = \{1, 2\}, S_2 = \{2, 3\}, S_3 = \{3, 4\}, S_4 = \{4, 5\}$ and $S_5 = \{1, 5\}$;

(ii) $S_1 = \{1, 2\}, S_2 = \{2, 3\}, S_3 = \{3, 4, 5\}, S_4 = \{1, 3\}$ and $S_5 = \{1, 2, 3\}$.

Does the family $(S_1, S_2, \cdots, S_5)$ have an SDR?

(i) The family $(S_1, S_2, \cdots, S_5)$ has an SDR, for instance, (1, 2, 3, 4, 5).

(ii) The family $(S_1, S_2, \cdots, S_5)$ does not have any SDR. Why? One way to argue is as follows: Observe that

$$S_1 \cup S_2 \cup S_4 \cup S_5 = \{1, 2, 3\}.$$

As there are more sets (4) than members (3 only), it is clear that no SDR for the family could exist.

In general, given a family of m sets $(S_1, S_2, \cdots, S_m)$, if there exist k of them, where $1 \le k \le m$, whose union only has less than k members, then it is obvious that the family does not have any SDR. That is, if there exists some $I \subseteq \{1, 2, \cdots, m\}$, I non-empty, such that

$$\left|\bigcup_{i\in I} S_i\right| < |I|,$$

then the family does not have any SDR. In other words, if $(S_1, S_2, \cdots, S_m)$ has an SDR, then

$$\left|\bigcup_{i\in I} S_i\right| \ge |I|,$$

for any subset I of $\{1, 2, \cdots, m\}$.

Is the converse true? That is, if $|\bigcup_{i\in I} S_i| \ge |I|$ for any subset I of $\{1, 2, \cdots, m\}$, is it true that the family $(S_1, S_2, \cdots, S_m)$ would have an SDR?

30 Hall's Theorem on SDR

The answer to the above question is in the affirmative, and this positive answer, as shown below, was given by Hall [3].

Theorem 30.1 *Let U be a non-empty set, and let $(S_1, S_2, \cdots, S_m)$ be a family of non-empty finite subsets of U, where $m \ge 1$. Then the family $(S_1, S_2, \cdots, S_m)$ has an SDR if and only if*

$$|\bigcup_{i\in I} S_i| \ge |I|,$$

for any subset I of $\{1, 2, \cdots, m\}$.

Exercise 30.1 *Applying Theorem 30.1, determine, in each of the following cases, if the family of sets of integers has an SDR:*

(i) $S_1 = \{1\}, S_2 = \{1, 2\}, S_3 = \{1, 2, 3\}, S_4 = \{1, 2, 3, 4\}$ *and* $S_5 = \{1, 2, 3, 4, 5\}$;

(ii) $S_1 = \{1, 2\}, S_2 = \{2, 3\}, S_3 = \{3, 4\}, S_4 = \{4, 1\}$ *and* $S_5 = \{2, 4\}$.

The necessity of Theorem 30.1, as pointed out earlier, is trivial. We shall now apply Hall's matching Theorem (Theorem 27.1) to prove the sufficiency of Theorem 30.1.

Proof of the sufficiency of Theorem 30.1.

Thus, assume that $(S_1, S_2, \cdots, S_m)$ is a given family of non-empty finite subsets of U, where $m \geq 1$, satisfying the condition that

$$|\bigcup_{i \in I} S_i| \geq |I|,$$

for any subset I of $\{1, 2, \cdots, m\}$. We shall show that the family $(S_1, S_2, \cdots, S_m)$ has an SDR.

To begin with, we form a bipartite graph G with bipartition (X, Y), where

$$X = \{S_1, S_2, \cdots, S_m\} \text{ and } Y = U,$$

such that S_i in X and y in Y are *adjacent* in G when and only when y is an *element* in S_i. (For instance, the bipartite graph G with bipartition (X, Y) corresponding to Example 29.1 is shown in Figure 30.1. Note that Y is the set of staff members in the department.)

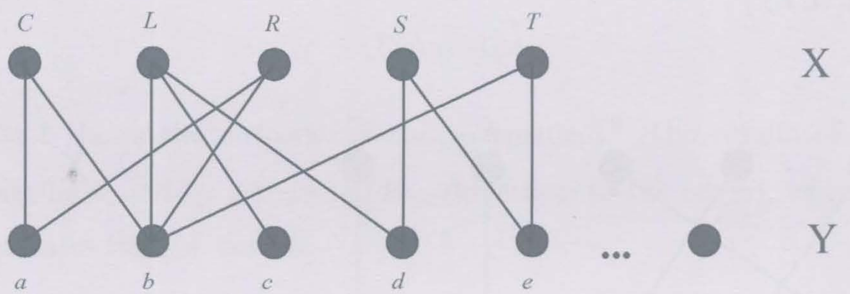

Figure 30.1

Next, we shall show that the inequality $|A| \leq |N(A)|$ holds in G for every subset A of X. Thus, let A be a subset of $X = \{S_1, S_2, \cdots, S_m\}$. We may write $A = \{S_i | i \in I\}$ for some subset I of $\{1, 2, \cdots, m\}$. Note that $|A| = |I|$.

We may ask: what is $N(A)$ in this case? Well, by our definition of G, $N(A)$ consists of all elements of S_i, where i is in I; that is,

$$N(A) = \bigcup_{i \in I} S_i.$$

Since the family $(S_1, S_2, \cdots, S_m)$ satisfies the condition that $|\bigcup_{i\in I} S_i| \geq |I|$, for any subset I of $\{1, 2, \cdots, m\}$, we then have

$$|N(A)| = |\bigcup_{i\in I} S_i| \geq |I| = |A|;$$

that is, $|A| \leq |N(A)|$ holds in G for every subset A of X, as required.

Accordingly, by Theorem 27.1, G possesses a complete matching from X to Y. Write this matching as

$$\begin{array}{c} S_1 - y_1, \\ S_2 - y_2, \\ \vdots \\ S_m - y_m. \end{array}$$

It is now clear that the sequence $(y_1, y_2, \cdots, y_m)$ is an SDR for the family $(S_1, S_2, \cdots, S_m)$. (For the bipartite graph G shown in Figure 30.1, a complete matching from X to Y exists, and is shown in Figure 30.2, which in turn produces an SDR, namely, (b, c, a, d, e) for the family (C, L, R, S, T).) □

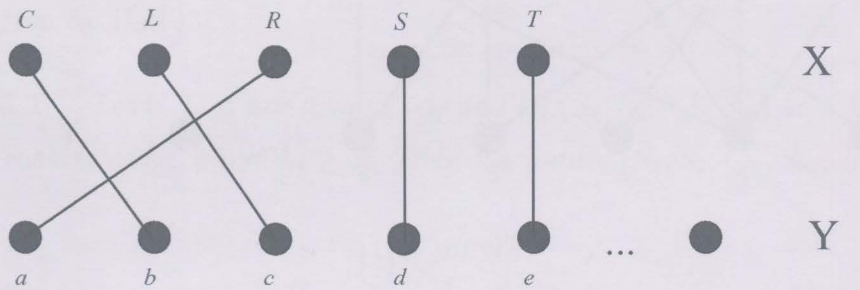

Figure 30.2

Exercise 30.2 *Find the number of SDR's for each of the following families, where n is a positive integer:*

(i) $\{1\}, \{1, 2\}, \{1, 2, 3\}, \cdots, \{1, 2, 3, \cdots, n\}$;

(ii) $\{1, 2\}, \{2, 3\}, \{3, 4\}, \cdots, \{n-1, n\}, \{n, 1\}$;

(iii) $\{1, 2\}, \{1, 3\}, \{1, 4\}, \cdots, \{1, n\}$;

(iv) $\{1, n+1\}, \{2, n+2\}, \cdots, \{n, 2n\}$.

31 Application - Score Sequences of Tournaments

Five tennis players B, D, F, N and R are invited to take part in a 5-man *round-robin tournament*, where any two of them engage in one and only one match that cannot end in a tie. The situation is modeled as the complete graph of order 5 of Figure 31.1, where the vertices represent the players and an edge joining 2 vertices denotes the game played by the 2 respective players.

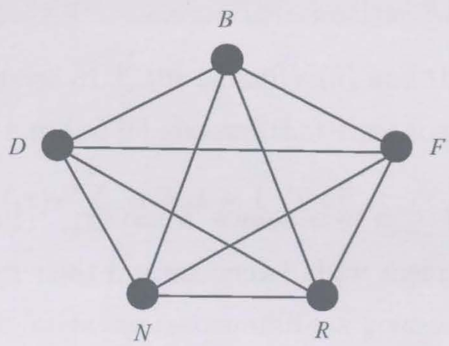

Figure 31.1

Figure 31.1 shows the outcome of the tournament. The results of the $\binom{5}{2}$ (= 10) games are indicated by arrows (adding directions to the edges), where, for instance, '$F \to N$' indicates that 'F defeats N'.

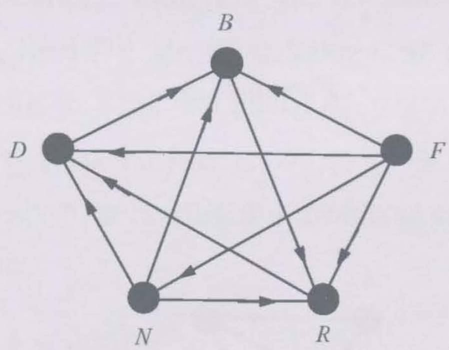

Figure 31.2

Mathematically, we have:

> A **tournament** is a non-empty finite set of vertices in which every 2 vertices are joined by one and only one arrow.

Let T be a tournament, and x, y be vertices in T. If the arrow joining x and y is **from** x **to** y in T, we say that 'x defeats y' or 'x dominates y'. The **score** of x, denoted by $s(x)$, is defined as the number of vertices in T dominated by x. Thus, in the tournament of Figure 31.2, we have:

$$s(B) = 1, s(D) = 1, s(F) = 4, s(N) = 3, \text{ and } s(R) = 1.$$

Exercise 31.1 *Suppose that T is a tournament with $n \geq 2$ vertices. Show that:*

(i) $0 \leq s(x) \leq n-1$, *for each vertex x in T;*

(ii) $\sum_{x \in T} s(x) = \binom{n}{2}$;

(iii) for any k vertices $x_1, x_2, \cdots, x_k$ in T, $1 \leq k \leq n$, $\sum_{i=1}^{k} s(x_i) \geq \binom{k}{2}$.

Figure 31.3 shows a tournament with 6 vertices and their respective scores.

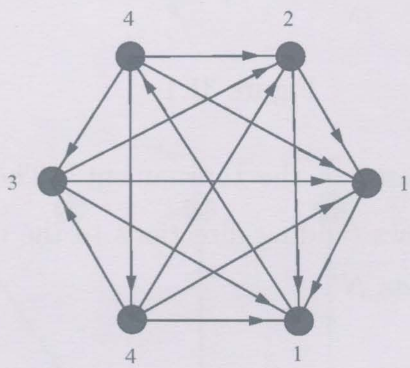

Figure 31.3

We may name the vertices as $v_1, v_2, \cdots, v_6$ so that $s(v_1) \leq s(v_2) \leq s(v_3) \leq s(v_4) \leq s(v_5) \leq s(v_6)$. One such naming is shown in Figure 31.4.

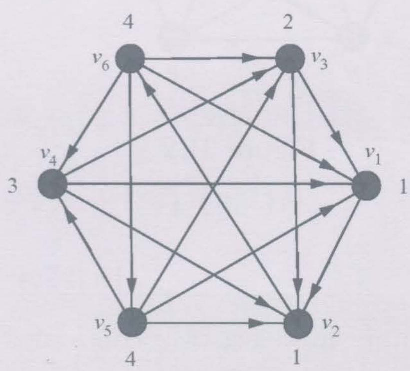

Figure 31.4

In general, let T be a tournament with $n \geq 2$ vertices $v_1, v_2, \cdots, v_n$ such that $s(v_1) \leq s(v_2) \leq \cdots \leq s(v_n)$. We call the sequence $(s(v_1), s(v_2), \cdots, s(v_n))$ the **score sequence** of T. Thus, (1, 1, 2, 3, 4, 4) is the score sequence of the tournament of Figure 31.3.

One related basic problem is the following:

> Given a sequence of n integers $(s_1, s_2, \cdots, s_n)$, where $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq n-1$, find necessary and sufficient conditions for the s_i's so that $(s_1, s_2, \cdots, s_n)$ is the score sequence of some tournament with n vertices.

By the results in Exercise 31.1, the equality (ii) and the inequality (iii) are two necessary conditions. Indeed, Landau [6] showed that they together are also sufficient.

Theorem 31.1 (Landau) *Given a sequence of $n \geq 1$ integers $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq n-1$, the sequence $(s_1, s_2, \cdots, s_n)$ is the score sequence of some tournament with n vertices if and only if the following two conditions hold:*

(i) for any k with $1 \leq k \leq n$, $\sum\limits_{i=1}^{k} s_i \geq \binom{k}{2}$; and

(ii) $\sum\limits_{i=1}^{n} s_i = \binom{n}{2}$.

Landau's Theorem is so 'great' that many researchers have found it worthwhile to find fresh proofs for it (its sufficiency). Until now, the theorem has received at least 10 different proofs (see, for instance, Reid [7]). In what follows, we shall introduce by example the idea of an elegant proof, due to Bang and Sharp [1], which makes use of **Hall's Theorem on SDR**.

Thus, suppose we are given a sequence of n integers $(s_1, s_2, \cdots, s_n)$ with $0 \leq s_1 \leq s_2 \leq \cdots \leq s_n \leq n-1$ satisfying

(i) for any k with $1 \leq k \leq n$, $\sum\limits_{i=1}^{k} s_i \geq \binom{k}{2}$ and

(ii) $\sum\limits_{i=1}^{n} s_i = \binom{n}{2}$.

Take, for example, the sequence $(s_1, s_2, \cdots, s_5) = (1, 2, 2, 2, 3)$ with $n = 5$ that satisfies (i) and (ii). The objective here is to introduce a general method which guarantees the existence of a tournament with 5 vertices that has (1, 2, 2, 2, 3) as its score sequence.

Step 1. For $i = 1, 2, \cdots, n$, let A_i be an arbitrary set with $|A_i| = s_i$, where the A_i's are pairwise disjoint. For our instance, let $A_1 = \{a\}, A_2 = \{b, c\}, A_3 = \{d, e\}, A_4 = \{f, g\}$ and $A_5 = \{p, q, r\}$.

Step 2. Form $A_i \cup A_j$, for all $1 \leq i < j \leq n$ (there are $\binom{n}{2}$ such unions). In our case, we have:

$$\begin{array}{ll} S_1 = A_1 \cup A_2 = \{a, b, c\}, & S_2 = A_1 \cup A_3 = \{a, d, e\}, \\ S_3 = A_1 \cup A_4 = \{a, f, g\}, & S_4 = A_1 \cup A_5 = \{a, p, q, r\}, \\ S_5 = A_2 \cup A_3 = \{b, c, d, e\}, & S_6 = A_2 \cup A_4 = \{b, c, f, g\}, \\ S_7 = A_2 \cup A_5 = \{b, c, p, q, r\}, & S_8 = A_3 \cup A_4 = \{d, e, f, g\}, \\ S_9 = A_3 \cup A_5 = \{d, e, p, q, r\}, & S_{10} = A_4 \cup A_5 = \{f, g, p, q, r\}. \end{array}$$

Step 3. Given that the sequence $(s_1, s_2, \cdots, s_n)$ satisfies the conditions (i) and (ii), Bang and Sharp showed that the family $(S_1, S_2, \cdots, S_m)$, where $m = \binom{n}{2}$, satisfies the inequality:

$$|\bigcup_{i \in I} S_i| \geq |I|,$$

for any subset I of $\{1, 2, \cdots, m\}$. Thus, by **Hall's Theorem on SDR**, the family $(S_1, S_2, \cdots, S_m)$ has an SDR. In our case, we have, for instance,

$$S_1 - b, S_2 - a, S_3 - f, S_4 - p, S_5 - c, S_6 - g, S_7 - q, S_8 - e, S_9 - d, S_{10} - r.$$

Step 4. Construct a tournament with n vertices $A_1, A_2, \cdots, A_n$ as follows: given $1 \leq i, j \leq n$, draw an arrow from A_i to A_j when and only when the representative of $A_i \cup A_j$ in the SDR obtained in Step 3 is from A_i. Thus, for our instance, the tournament with 5 vertices is shown in Figure 31.5.

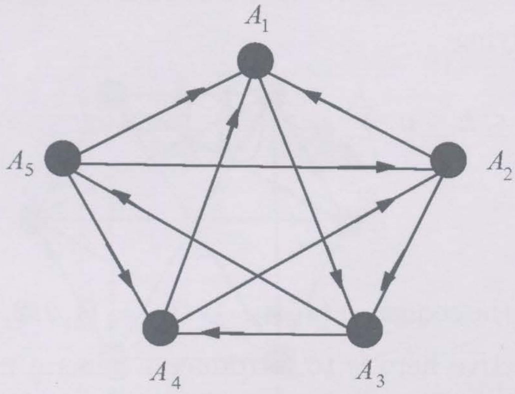

Figure 31.5

Bang and Sharp showed that this tournament has its score sequence equal to the given sequence $(s_1, s_2, \cdots, s_n)$. $\square$

Exercise 31.2 *Apply Landau's Theorem to determine whether each of the following sequences is the score sequence of some tournament. If your answer is 'yes', then apply Bang and Sharp's method to construct such a tournament.*

(i) $(1, 1, 2, 3, 3)$; *(ii)* $(1, 1, 1, 2, 5, 5)$.

Further Remarks

In Exercise 30.2, the reader is asked to find the number of SDR's for some families of subsets. In general, the number of SDR's of a family of n subsets $R_1, R_2, \cdots, R_n$ of $\{1, 2, \cdots, n\}$ can be found directly from the *permanent* of the square matrix $A = (a_{ij})$, where the rows represent the subsets and $a_{ij} = 1$ if $j \in R_i$, otherwise $a_{ij} = 0$. The permanent of A, $per(A)$, is given by

$$per(A) = \sum_{\sigma} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where σ is a permutation of $(1, 2, \cdots, n)$ and the summation is taken over all $n!$ permutations. The study and calculation of permanents itself is an important topic in research (see for example, Ryser [8] or Jerrum, Sinclair and Vigoda [4]).

Optimum transmission of data across a computer network is very important in today's highly-wired world. Interestingly, a generalisation of a system of distinct representatives has been proposed by Gao, Novick and Qiu [2] to reduce by half the delay time in the transmission of data across a computer network modeled on an n-hypercube. They suggest the use of *disjoint orderings*. A permutation of the elements of a finite set is called an ordering. Suppose X and Y are two sets ordered as $O_1 = (x_1, x_2, \cdots, x_k)$ and $O_2 = (y_1, y_2, \cdots, y_l)$, where $k = |X|$ and $l = |Y|$. O_1 and O_2 are said to be disjoint if for every $1 \leq t \leq min(k, l)$, $\{x_1, x_2, \cdots, x_t\} \neq \{y_1, y_2, \cdots, y_t\}$ as sets, unless $t = k = l$. A collection of finite sets is said to have a disjoint ordering if each set has an ordering such that all the orderings are pairwise disjoint. The concept of a disjoint ordering is a generalization of a system of distinct representatives. In the first place, if all singletons in a collection are distinct, the first elements of each ordering in a disjoint ordering will form a system of distinct representatives. Secondly and surprisingly, Hall's marriage condition is also a necessary and sufficient condition for the existence of a disjoint ordering.

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