Discrete Mathematics in the game of Set

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1 Introduction to The Original Game

To avoid confusion we distinguish between the words SET*, Set and set: SET*: Name of the game Set: a collection of three cards that a player must find in the game set: set in the usual, mathematical sense.

In the game SET*, each card has some symbols drawn on it. The symbols have four attributes, namely: colour, shape, number and shading. In terms of colour, the symbol can be green, red or purple. In terms of shape, it can be a diamond, oval or squiggle. In terms of number, there can be 1, 2 or 3 symbols. In terms of shading, the symbol can be blank, partially shaded or fully shaded. We say that there are 3 possible values for each attribute. The deck consists of different cards so that each card appears exactly once in the deck. There are several different forms of playing the game, but the objective of the game is always to find the most number of Sets. A Set is a collection of 3 cards such that for each attribute, the values are either all same or all different.

2 Objectives

In our project, we generalise the game to allow arbitrary number of attributes and arbitrary values for each attribute. With this generalized setting, we investigate the mathematical aspects of the generalised game. Through doing so, we hope to make learning discrete mathematics more interesting.

3 Generalisation / Representation

Let A be the number of attributes and V be the number of values for each attribute. We also want the deck to contain exactly one card for each possible combination of values of attributes, so the number of cards is V^A.

To represent the value of one attribute, we can associate each of the V values for an attribute to an element of the set $\mathbb{Z}_V$. The set $\mathbb{Z}_v = \{0, 1, 2, \ldots, V-1\}$ is a group under the addition modulo V. To represent attributes, we can use a tuple of length A with each of the entries being an element of $\mathbb{Z}_V$. We denote the group of all such tuples by $\mathbb{Z}_V^A$. We define the group operation $+$ as follows: For every $(x_1, \ldots, x_A), (y_1, \ldots, y_A) \in \mathbb{Z}_V^A$,

$$(x_1, \ldots, x_A) + (y_1, \ldots, y_A) = (x_1 + y_1, \ldots, x_A + y_A).$$

The above representation is adapted from the representation found in the article "The Joy of Set" by Mike Zabrocki.

Next, we aim to define a Set in the general game. There are many possible ways to generalize the definition of a Set. We have chosen the following two definitions:

Definition 1. V cards form a Set if and only if for every attribute, the values of all the V cards are either all the same or all different.

Definition 2. A Set is a collection of V distinct points $C_1, C_2, \ldots, C_v \in \mathbb{Z}_V^A$ such that

$$C_1 + C_2 + \cdots + C_v = (0, 0, \ldots, 0).$$

The first definition may seem more intuitive to players of the original game, while the second definition relates to our representation of cards by elements in $\mathbb{Z}_V^A$. We tried relating the two definition and we found that the two definitions are equivalent only when $|V| = 3$. Also, when V is odd, a Set under the first definition is also a Set under the second definition though the converse is not necessarily true.

4 Key Problems

Question 1. *What is the probability that in a Set under the first definition, r properties are different and $A - r$ properties are the same?*

To solve this problem, we first need to find the number of Sets with properties are different and $A - r$ properties are the same and the total number of Sets in the deck under the first definition.

Let us first find the former. There are $\binom{A}{r}$ ways of choosing properties having different values. Of the $A-r$ properties having the same value, there are V^{A-r} possible combinations as each of the $A-r$ properties have V possible choices. Of the r properties with different values, there are $V!$ combinations for each property, so there are $(V!)^r$ possible combinations for the properties with different values. However, we have counted each Set $V!$ times. By multiplication principle, there are $\binom{A}{r}V^{A-r}(V!)^{r-1}$ Sets in a deck with exactly r properties having the same value.

Now let us find the total number of Sets in the deck under the first definition. The total number of Sets in a deck is the sum of the number of Sets with r attributes different for all r from 1 to A (r cannot be 0 because the cards are distinct), which is given by the following expression:

$$\sum_{r=1}^{A}\binom{A}{r}V^{A-r}(V!)^{r-1}.$$

Since each of the Sets is equally likely to appear, the answer to our problem is given by

$$\binom{A}{r}V^{A-r}(V!)^{r-1}/[\sum_{i=1}^{r}\binom{A}{i}V^{A-i}(V!)^{i-1}].$$

Question 2. *What is the total number of Sets in a deck under the second definition?*

This is equivalent to the total number of combinations of V distinct elements of $\mathbb{Z}_V^A$ adding up to $(0,\ldots,0)$.

Let U be the set of all combinations of V elements of $\mathbb{Z}_V^A$ adding up to $(0,\ldots,0)$. Let P be the set of all combinations consisting of V *distinct* elements of $\mathbb{Z}_V^A$ adding up to $(0,\ldots,0)$. The cardinality of P, denoted by $|P|$ is what we want to find.

Let Q be the set of all combinations of V elements of $\mathbb{Z}_V^A$ such that the V elements add up to $(0,\ldots,0)$ and at least two of the elements are the same in each combination. Thus, $|P|=|U|-|Q|$.

Let S_i be the set of all combinations of V elements of $\mathbb{Z}_V^A$ such that the V elements add up to $(0,\ldots,0)$ and at least two out of the V elements are C_i in each combination. Observe that

$$Q=\bigcup_{i=1}^{V^A}S_i.$$

We shall find $|Q|$ by using the Principle of Inclusion-Exclusion. The principle of Inclusion-Exclusion states that: If $A_1,\ldots,A_n$ are finite sets, then

$$\left|\bigcup_{i=1}^{n}A_i\right|=\sum_{i=1}^{n}|A_i|-\sum_{i<j}|A_i\cap A_j|+\sum_{i<j<k}|A_i\cap A_j\cap A_k|-\cdots+(-1)^{n-1}|A_1\cap A_2\cdots A_n|.$$

Let $x_1, \ldots, x_n$ be n distinct integers such that $1 \leq x)i \leq V^A$. Consider the intersection of the sets $S_{x_1}, \ldots, S_{x_n}$. It is the set of all combinations of V elements in $\mathbb{Z}_V^A$ with at least two of C_{x_1}, two of C_{x_2} etc.

Hence $2n$ out of V elements in each combination are determined already. (This implies that $n \leq [V/2]$, where $[V/2]$ is the greatest integer smaller than or equals to $V/2$.)

Given $V-1$ cards, $C_1, \ldots, C_{V-1}$, the V-th card, $C_V = -\sum_{i=1}^{V-1} C_i$ and adding all V cards add up to $(0, 0, \ldots, 0)$. Therefore, the last card is uniquely determined by the first $V-1$ cards. Hence we only need to choose another $V-2n-1$ cards. The number of ways of getting a combination of k objects from n distinct objects with repetitions is given by $\binom{n+k-1}{k}$. Hence for each of the $\binom{V^A}{n}$ possible $x_1.x_2, \ldots, x_n$,

$$\left|\bigcap_{i=1}^{n} S_{x_i}\right| = \binom{V^A+V-2n-2}{V-2n-1}.$$

Note that $|S_{x_i}|$ corresponds to the case when $n=1$. By the Principle of Inclusion-Exclusion,

$$|Q| = \sum_{i=1} [V/2](-1)^{i+1}\binom{V^A}{i}\binom{V^A+V-2i-2}{V-2i-1}.$$

Now let us find the cardinality of U. We need to choose $V-1$ elements of $\mathbb{Z}_V^A$, with V^A choices, giving us $\binom{V^A+V-2}{V-1} = |U|$. Therefore,

$$\begin{aligned}
|P| = |U| - |Q| &= \binom{V^A+V-2}{V-1} + \sum_{i=1}^{[V/2]}(-1)^i\binom{V^A}{i}\binom{V^A+V-2i-2}{V-2i-1} \\
&= \sum_{i=1}^{[V/2]}(-1)^{i+1}\binom{V^A}{i}\binom{V^A+V-2i-2}{V-2i-1}.
\end{aligned}$$

Question 3. What is the maximum number of cards in the general game not containing a Set?

In the article "The Joy of Set" by Mike Zabrocki, it was suggested that this might be linked to the Ramsey Theory. We have tried to relate it to Ramsey Theory, but to no avail.

5 Conclusion

We have come up with two forms of the generalised game and linked the generalised game with many areas of discrete mathematics such as modular arithmetic, counting, probability and the principle of inclusion-exclusion in both our representation and problems. However, because of the wide range of questions that arises from the generalised game, we have left some problems unsolved and can be explored in future. The game can be generalised in other ways and its properties may be interesting. Much of the probabilistic aspects of the game can be further explored, such as the probability of getting a certain number of Sets within a certain number of cards in the general game. There are also various ways of playing the game, and this may be analysed using game theory.

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