PSLE 2005 and Curry's Paradox

by
Helmer Aslaksen
Department of Mathematics
National University of Singapore

aslaksen@math.nus.edu.sg
www.math.nus.edu.sg/aslaksen/

1 PSLE 2005

Question 13 on the PSLE Mathematics Exam in Singapore in 2005 was as follows ([3]). The figure below shows a rectangle 15 cm by 6 cm. The area of Triangle A is 4 cm^2 and the area of Triangle B is 16 cm^2.

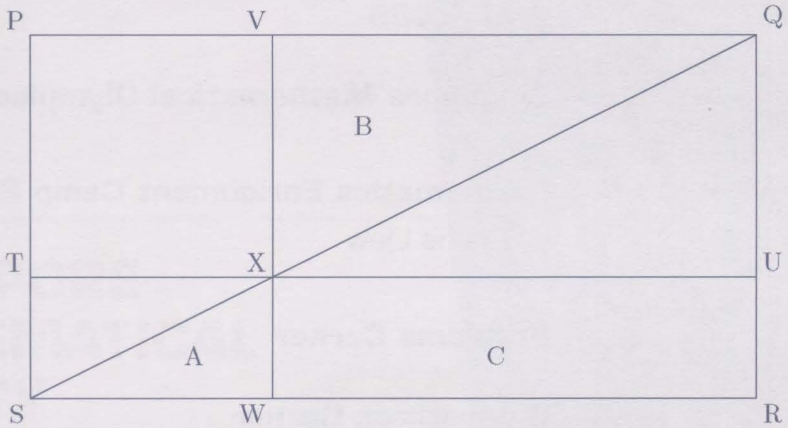

Figure 1: Question 13 on the PSLE Mathematics Exam

What is the area of Rectangle C?

(1) 20 cm^2 (2) 22 cm^2 (3) 25 cm^2 (4) 28 cm^2

The examiners expected the students to argue that the area of triangle SQR is $6 \times 15/2 = 45$, and that the area of rectangle C therefore is $45 - 16 - 4 = 25$.

However, some students noticed that the numbers did not match up. The easiest way to see the problem is to notice that X lies on the diagonal SQ. In that case the sides of the rectangle are cut in the same ratio, and since the ratio of the areas of the two small triangles is 4, the sides are cut in the ratio of 2 to 1. In other words, $a = 4$ and $b = 10$, in which case the area of C is 20 and not 25. However, the areas of the triangles are then 5 and 20 instead of 4 and 16, so there is clearly a problem.

Another way of looking at this is that at an exam at this level we usually expect the length of the sides to be integers. Some quick guessing would suggest that ST is 2 and SW is 4, in which the area of triangle A is 4 as stated. However, in that case WR would be 11, which looks a bit odd (no pun intended!). It is not too hard to see that there is no way to find integer lengths of the sides of the subrectangles that match the given areas. (Provided we assume that X lies on the diagonal SQ.)

Several students noticed that both option (1) and (3) seemed to be possible answers and were confused about it during the exam. One student asked a teacher in charge about this, and was told there was nothing wrong with the exam. The Singapore Examination and Assessment Board later admitted that the question was flawed and said "it is mathematically not possible to draw such a figure".

The question is probably a variation of another problem. The person who wrote the question was probably trying to simply change the numbers and did not realize the geometric constraints. In fact, it is quite easy to change the question to a correct one. If we make the long side SR 12, then the short side SP can be cut into 2 and 4 and the long side SR can be cut into 4 and 8, in which case the areas of the triangles would be 4 and 16 as stated.

People involved in education know that it is surprisingly hard to set a an exam without any ambiguities or errors. Fortunately, most of the time the problems are so subtle that nobody notices or some common sense will make it obvious what was meant. It is also hard to vet exams. I cannot promise that I would have caught this problem if it had been my job to vet the exam.

The point of this note is not to dwell on an unfortunate accident. However, this question is quite subtle and raises several interesting issues of interest to mathematicians and educators. In Section 2 we will discuss alternative interpretations of the question and in Section 3 we will describe the link with Curry's Paradox.

2 Alternative interpretations

The intended solution makes the crucial assumption that X lies on the diagonal SQ. Otherwise the two triangles and rectangle C would not form a triangle.

But how do we know that X lies on a diagonal? We are not told explicitly so. Some may argue that it follows from the picture, but we will see in Section 3 that

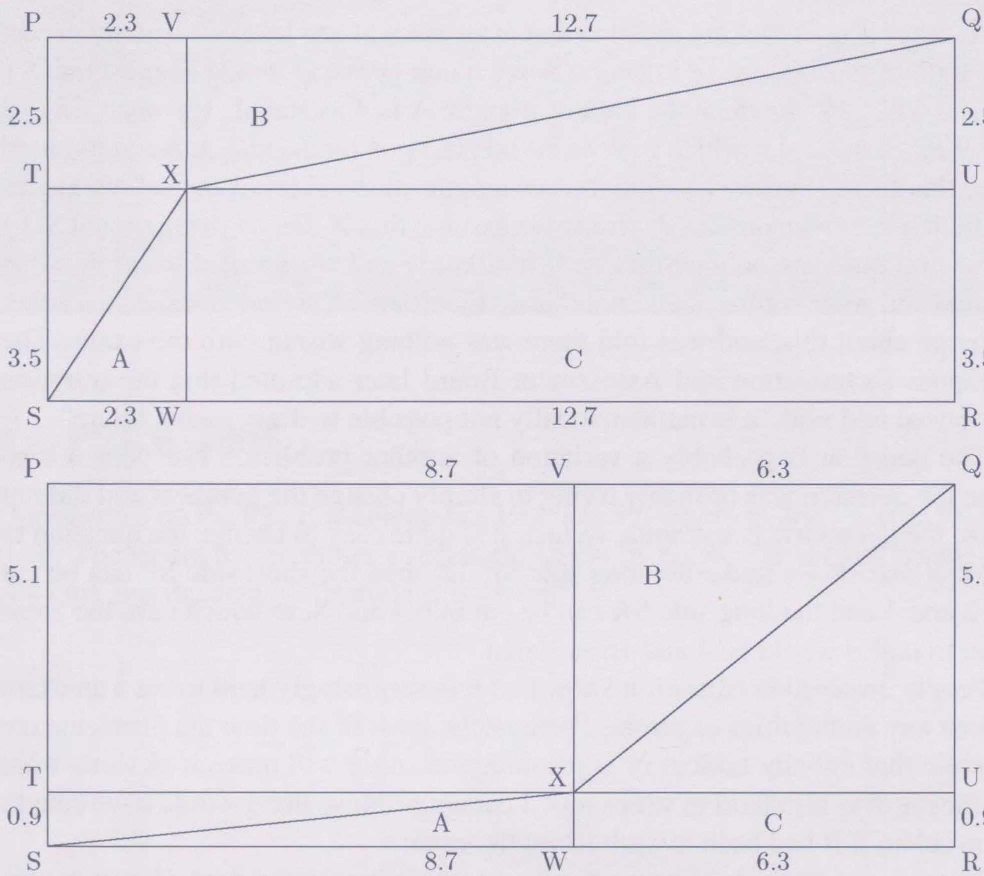

Figure 2: The two possible figures where X is not on a diagonal

we cannot always trust our intuition in cases like that.

From a mathematical point of view it makes perfect sense to consider a subdivision of the rectangle into four subrectangles where we do not assume that X lies on the diagonal. If we set PT $= a$ and VQ $= b$, then $ab/2 = 16$ and $(6-a)(15-b)/2 = 4$. This gives $b = 32/a$ and $15a^2 - 114a - 192 = 0$ and we get two solutions

$$a = (19-\sqrt{41})/5 \approx 2.5, \quad b = (19+\sqrt{41})/2 \approx 12.7 \quad \text{or}$$
$$a = (19+\sqrt{41})/5 \approx 5.1, \quad b = (19-\sqrt{41})/2 \approx 6.3.$$

These two cases are illustrated in Figure 2. In the first case, the area of rectangle C equals $b(6-a) \approx 44.4$, while in the second case $b(6-a) \approx 5.8$.

Another issue is whether the lines TU and VW are parallel to the sides of the rectangle. This is the same as requiring that triangles A and B are right triangles. If they are not, the rectangle will be cut into four quadrilaterals and C will not be a rectangle.

The question explicitly describes C as a rectangle, so we do not need to consider this case. We will just show one example of a subdivision of the rectangle into four quadrilaterals where the area of the triangles A and B are 4 and 16. In the example in Figure 3 we made TU parallel to SR to make it easier to calculate the area of the triangles.

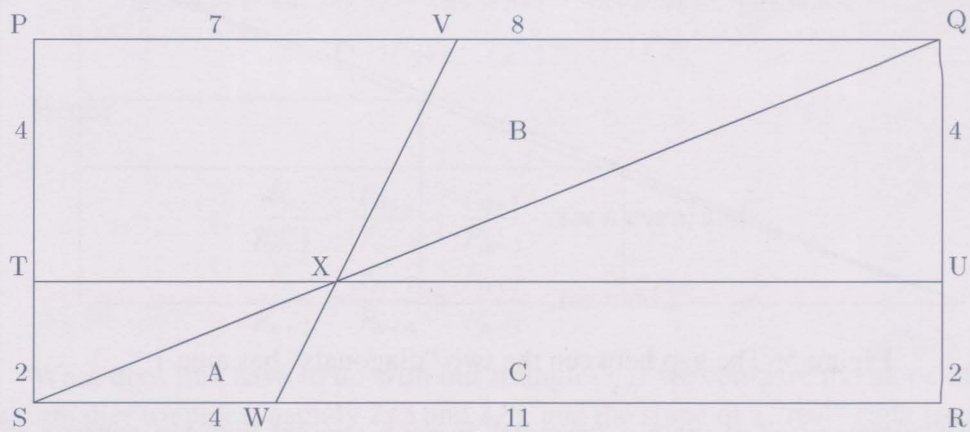

Figure 3: A figure where the triangles are not right triangles

3 Curry's Paradox

The question of whether the intersection point lies on the diagonal or not is related to the famous Curry's Paradox ([2, 5]), due to a New York City amateur magician.

Some years ago this was used as part of an advertising campaign on the MRT in Singapore, so it may be familiar to many people.

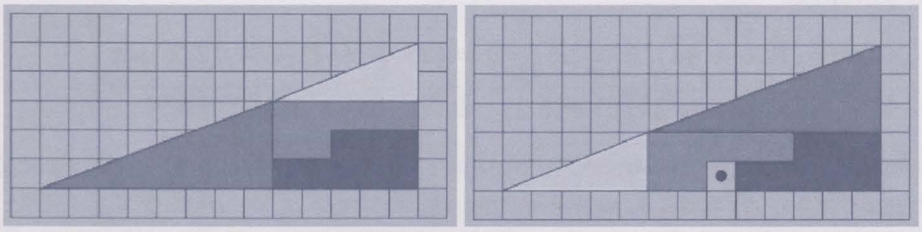

Figure 4: Graphics by Eric W. Weisstein ([6])

Both parts of Figure 4 seem to be dissections of a right triangle by two smaller right triangles and a rectangle, except that the second rectangle requires an extra unit. The bottom right corner is a 3×5 rectangle in the first case and a 2×8 rectangle in the second case. The explanation for this paradox is that the big triangle is not a triangle!

Figure 5: The gap between the two "diagonals" has area 1

The "hypotenuse" of the big "triangle" is not a straight line, but consists of two broken segments. As a result, the "hypotenuse" of the left part of Figure 4 is slightly bent in, whereas the "hypotenuse" of the right part of Figure 4 is slightly bent out. This shows that we cannot always trust our intuition, and it may not always be correct to make assumptions like saying that X lies on a diagonal in the PSLE question.

The dimensions are not picked at random. The legs forming the right angles in the three triangles are $(5,2)$, $(8,3)$ and $(13,5)$, and we immediately see the Fibonacci patterns.

The Fibonacci numbers F_n are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$ and satisfy many interesting relations. One of them is Catalan's Identity, named after the Belgian mathematician Eugène Charles Catalan (1814–1894)

$$F_n^2 - F_{n+r}F_{n-r} = (-1)^{(n-r)}F_r^2.$$

If we set $r = 1$, we get the following identity, due to the French astronomer Jean-Dominique Cassini (1625–1712) in 1680, but also proved independently by Simpson in 1753

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

It follows from Catalan's Identity that

$$F_{n+2}^2 - F_{n+4}F_n = (-1)^n F_2^2 = (-1)^n$$

and

$$\begin{aligned} F_{n+4}F_{n+1} - F_{n+3}F_{n+2} = F_{n+3}F_{n+1} + F_{n+2}F_{n+1} - F_{n+3}F_{n+2} = \\ F_{n+3}F_{n+1} - F_{n+2}^2 = (-1)^n. \end{aligned}$$

Hence

$$\begin{aligned} &\frac{F_n}{F_{n+2}} < \frac{F_{n+2}}{F_{n+4}} < \frac{F_{n+1}}{F_{n+3}} \quad \text{for } n \text{ even, and} \\ &\frac{F_n}{F_{n+2}} > \frac{F_{n+2}}{F_{n+4}} > \frac{F_{n+1}}{F_{n+3}} \quad \text{for } n \text{ odd.} \end{aligned}$$

What does this have to do with our triangles? If we compare the slope of the two smaller triangles, namely $2/5$ and $3/8$, and the slope of a "real" right triangle with short sides $(13,5)$, we see that

$$\frac{2}{5} > \frac{5}{13} > \frac{3}{8}.$$

This shows that the we cannot combine the two smaller triangles to get a big triangle. However, since $26 - 25 = 1$ and $40 - 39 = 1$, we see that the fractions are close, so the two "triangles" look similar.

In the same way we can compare the areas under the two "triangles" and under the real $(13,5)$ triangle. Using Cassini's Identity we see that

$$\begin{gathered}
2F_{n+1}F_{n-2} - (F_{n+2}F_n - F_{n+1}F_{n-1} - F_nF_{n-2}) = \\
F_{n+1}F_{n-2} + F_nF_{n-2} + F_{n+1}F_{n-2} + F_{n+1}F_{n-1} - F_{n+2}F_n = \\
F_{n+2}F_{n-2} + F_{n+1}F_n - F_{n+2}F_n = \\
F_{n+2}F_{n-2} - F_{n+2}^2 = (-1)^{n+1}
\end{gathered}$$

and

$$\begin{gathered}
(F_{n+2}F_n - F_{n+1}F_{n-1} - F_nF_{n-2}) - 2F_nF_{n-1} = \\
F_{n+2}F_n - (F_{n+1}F_{n-1} + 2F_nF_{n-1}) - (F_nF_{n-2} + 2F_nF_{n-1}) = \\
F_{n+2}F_n - F_{n+2}F_{n-1} - F_n^2 = \\
F_{n+2}F_{n-2} - F_n^2 = (-1)^{n+1}.
\end{gathered}$$

For $n = 5$ we get

$$\begin{gathered}
3 \times 8/2 + 3 \times 5 + 2 \times 5/2 = 32, \\
5 \times 13 = 32.5, \\
2 \times 5/2 + 2 \times 8 + 3 \times 8/2 = 33.
\end{gathered}$$

The difference in the areas of these figures is exactly the "extra" one unit.

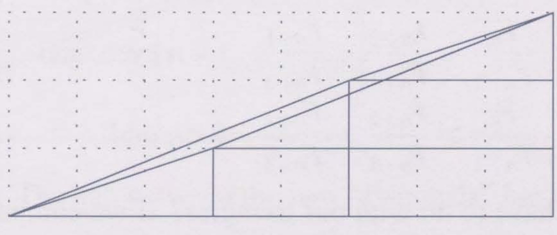

Figure 6: The gap between the two "diagonals" has area 1

Instead of using right triangles with short sides $(5,2)$ and $8,3)$, we could have used $(3,1)$ and $(5,2)$ as in Figure 6. However, in this case

$$\frac{2}{5} > \frac{3}{8} > \frac{1}{3}$$

is not as close an approximation as

$$\frac{2}{5} > \frac{5}{13} > \frac{3}{8},$$

so the visual effect is less convincing.

References

[1] Greg N. Frederickson, Dissections: Plane and Fancy, Cambridge University Press, 1997.

[2] Martin Gardner, Mathematics, Magic and Mystery, Dover Publication, 1956.

[3] HO Ai Li, MOE admits to error in PSLE maths paper, The Straits Times, Singapore, October 12, 2005.

[4] HO Ai Li, Open Question 13, The Sunday Times, Singapore, October 16, 2005.

[5] Torsten Sillke, Jigsaw Paradox, http://www.mathematik.uni-bielefeld.de/~sillke/PUZZLES/jigsaw-paradox.html.

[6] Eric W. Weisstein. "Triangle Dissection Paradox."

From MathWorld – A Wolfram Web Resource, http://mathworld.wolfram.com/TriangleDissectionParadox.html.