Graphs and Their Applications (7)

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20 Short Review

We introduced in [2] the notion $\tau(G)$, which is the number of spanning trees of a given multigraph G. Two methods of computing $\tau(G)$ were also presented therein.

The first method, which is recursive in nature, states that

$$\tau(G) = \tau(G - e) + \tau(G \cdot e)$$

for each edge $e = uv$ in G, where $G - e$ is the multigraph obtained from G by deleting 'e' and $G \cdot e$ is the multigraph obtained from G by first deleting all edges joining u and v, and then identifying u and v.

The second method, known as the Kirchhoff's Matrix-Tree Theorem, makes use of the notion 'matrix' as shown below (see Theorem 19.1 in [2]).

Kirchhoff's Matrix-Tree Theorem. For any multigraph G with $V(G) = \{v_1, v_2, \cdots, v_n\}$, $\tau(G)$ is equal to the cofactor of any entry in $D(G) - A(G)$, where $A(G) = (a_{i,j})$ is the *adjacency matrix* of G in which $a_{i,j}$ is the number of edges joining v_i and v_j and $D(G) = (d_{i,j})$ is the degree matrix of G defined by

$$d_{i,j} = \begin{cases} \text{the degree of } v_i, & \text{if } j = i; \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that $\tau(G) \geq 1$ if and only if G is connected; and for a connected multigraph G, $\tau(G) = 1$ if and only if G is a tree (see Exercise 17.1[2]). Let G be a connected multigraph. While $\tau(G) \geq 1$, we now look for a good upper bound for $\tau(G)$. Clearly, this problem does not make sense if there is no control on the number of parallel edges joining pairs of vertices. Let us thus confine the problem to (simple) graphs and ask: if G is a graph of order n, what is the largest value that $\tau(G)$ can attain? We note that if G is a spanning subgraph of a graph H, then $\tau(G) \leq \tau(H)$ (see Exercise 17.5[2]). It thus follows that $\tau(G) \leq \tau(K_n)$. Is there a formula for $\tau(K_n)$?

In this article, our objective is to present a celebrated result, known as Cayley's formula, which counts for $\tau(K_n)$ for all $n \geq 1$.

21 Cayley's Formula

In 1889, Arthur Cayley, the famous British mathematician, published his paper [1] in which he established the following beautiful result:

Theorem 21.1 *For each integer $n \geq 2$, $\tau(K_n) = n^{n-2}$.*

Arthur Cayley (1821 - 1895) helped found the modern British school of pure mathematics. He was a prolific mathematician and published about 900 research papers in his career.

For example, when $n = 4$, there are 4^{4-2} $(= 16)$ different spanning trees of K_4, as shown in Figure 21.1.

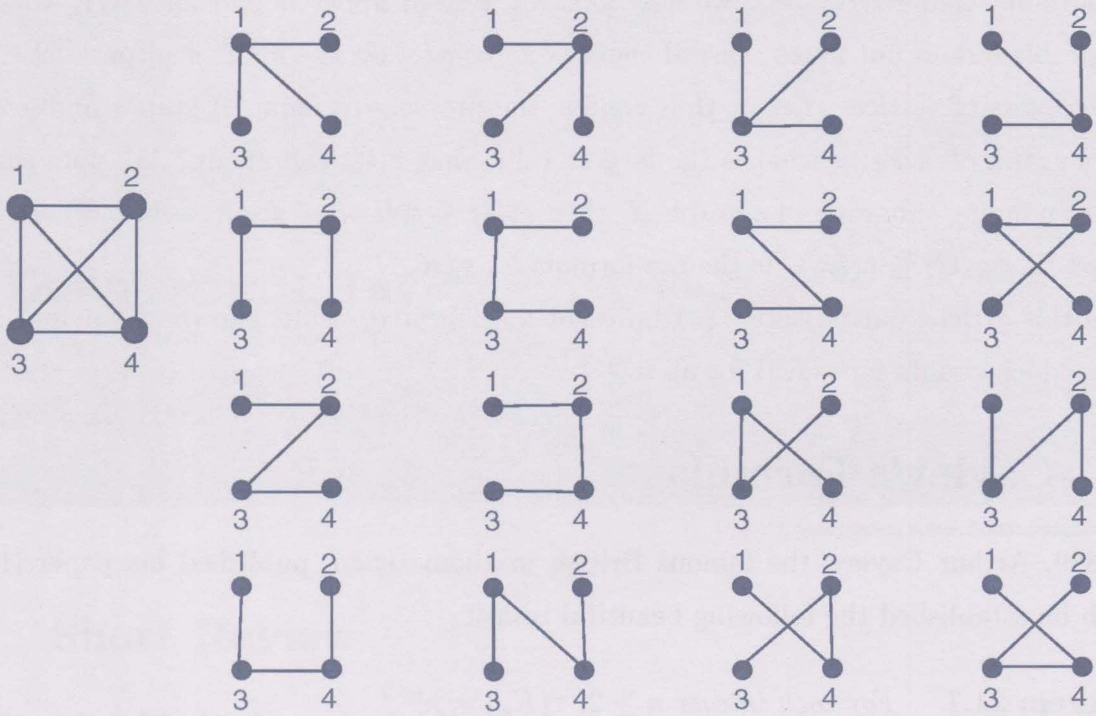


Figure 21.1

There are many different proofs of Cayley's formula. For instance, the Canadian mathematician John W. Moon showed ten of them in his paper [3], which was published almost 40 years ago. In this and the next section, we shall introduce, by example, the ideas of two proofs.

The first proof is by means of Kirchhoff's Matrix-Tree theorem. We shall consider the case when $n = 5$, and the reader will easily find that the argument can naturally be extended to the general case.

Thus, our objective now is to show that $\tau(K_5) = 5^3$ by applying Kirchhoff's theorem. Observe that

$$D(K_5) = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \text{ and } A(K_5) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Thus

$$D(K_5) - A(K_5) = \begin{pmatrix} 4 & -1 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & 4 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}.$$

By Kirchhoff's theorem,

$$\begin{aligned} \tau(K_5) &= \text{ the cofactor of the } (1,1) \text{ entry of } D(K_5) - A(K_5) \\ &= (-1)^{1+1} \begin{vmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{vmatrix} \\ &= \begin{vmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{vmatrix}. \end{aligned} \tag{21.1}$$

We now compute the above determinant by applying the properties of determinant as follows:

For $i = 2, 3, 4$, adding row i to row 1 successively on (21.1) results in

$$\tau(K_5) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{vmatrix}. \tag{21.2}$$

Now, for $i = 2, 3, 4$, adding row 1 to row i successively on (21.2) yields

$$\tau(K_5) = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{vmatrix} = 1 \times 5 \times 5 \times 5. \tag{21.3}$$

We thus arrive at $\tau(K_5) = 5^3$, as asserted.

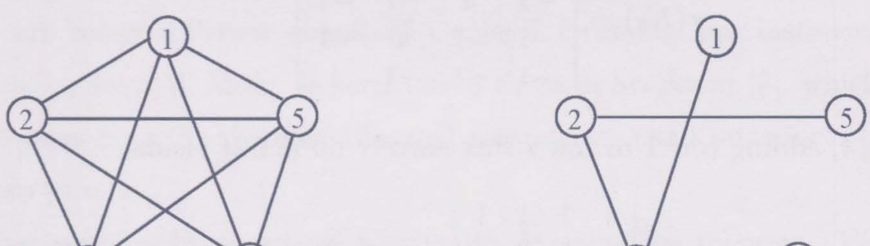
22 Prufer's Bijection

In this section, we shall prove Cayley's formula (for $n = 5$ also) by a beautiful combinatorial method due to Prufer [4].

The method is based on the Bijection Principle which has been used in the proof of Theorem 18.1. For convenience, denote the vertices of K_5 by $1, 2, 3, 4$ and 5. Let A be the set of all spanning trees of K_5 and B the set of all sequences (s_1, s_2, s_3) of length 3, where s_1, s_2, s_3 are in $\{1, 2, 3, 4, 5\}$ and are not necessarily distinct. The idea is to establish a one-one (and onto) correspondence between the set A and the set B which would then imply that $|A| = |B|$.

How many members (sequences (s_1, s_2, s_3)) are there in B? Observe that for $i = 1, 2, 3$, each s_i has 5 choices (namely, $1, 2, 3, 4$ or 5) independently. It follows that $|B| = 5 \cdot 5 \cdot 5 = 5^3$. Thus, the question now is: how to establish an appropriate one-one correspondence between A and B?

In 1918, H. Prufer published a short note [4] in which he gave a new and elegant proof of a result on permutations due to Herr Dziobek (1917). His proof (Prufer mentioned in [4] that the idea was suggested by Schur) indeed amounts to establishing such an interesting one-one correspondence between A and B. We now illustrate this idea for K_5.



K_5 A spanning tree T of K_5

Figure 22.1

Consider the spanning tree T shown in Figure 22.1. There are 3 end-vertices, namely $1, 4, 5$ (always in increasing order). The first one is '1'. Which vertex is '1' adjacent to in T? The answer is '3'. Let $s_1 = 3$.

Delete '1' from T and consider the resulting tree T' as shown in Figure 22.2:

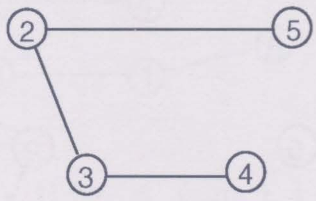


Figure 22.2

There are 2 end-vertices in T', namely $4, 5$. The first one is '4'. Which vertex is '4' adjacent to in T'? The answer is '3'. Let $s_2 = 3$.

Delete '4' from T' and consider the resulting tree T'' as shown in Figure 22.3:

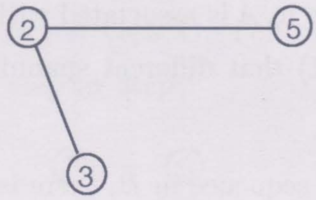


Figure 22.3

There are 2 end-vertices in T'', namely $3, 5$. The first one is '3', and it is adjacent to '2' in T''. Let $s_3 = 2$. Thus the spanning tree T of K_5 in Figure 22.1 is associated with the sequence $(s_1, s_2, s_3) = (3, 3, 2)$.

Consider the spanning tree T of K_5 shown in Figure 22.4:

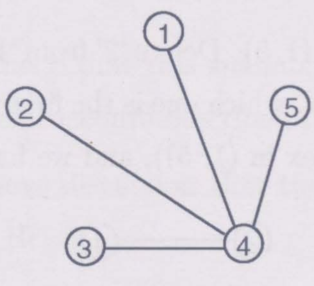


Figure 22.4

Following the same procedure, we see that T is associated with the sequence $(4, 4, 4)$. Some other examples are shown in Figure 22.5.

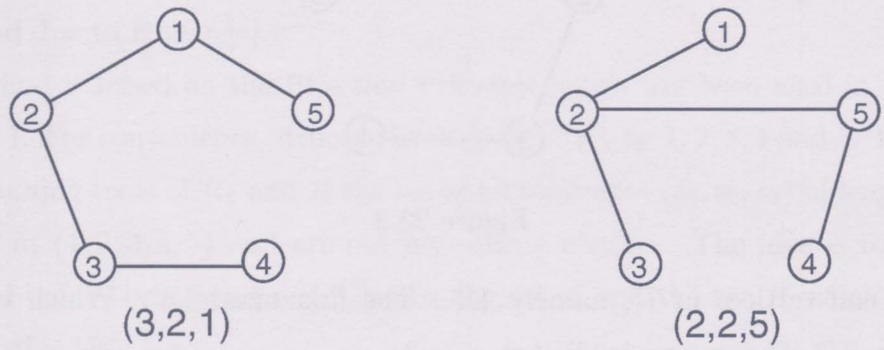
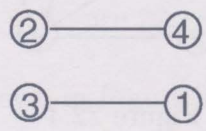


Figure 22.5

Thus, we see that every spanning tree in A is associated with a sequence in B, and it can be verified further (see Exercise 22.1) that different spanning trees in A are associated with different sequences in B.

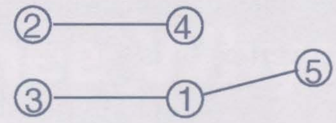
It remains to show that given any sequence in B, there is a spanning tree in A which is associated with it. Again, we illustrate it by example. Take a sequence, say, $(4, 1, 5)$ in B, which is regarded as the sequence of three vertices '4', '1', and '5'. Among the five vertices '1, 2, 3, 4, 5' of K_5, which one is the first vertex not in the sequence? The answer is '2'. Join '2' and '4' (the first vertex in the sequence) as shown below:

Delete '4' from $(4, 1, 5)$ resulting in $(1, 5)$. Delete '2' from '1, 2, 3, 4, 5' resulting in '1, 3, 4, 5'. Among these four vertices '1, 3, 4, 5', which one is the first vertex not in $(1, 5)$? The answer is '3'. Join '3' to '1' (the first vertex in (1, 5)), and we have:



Delete '1' from $(1, 5)$ resulting in (5). Delete '3' from '1, 3, 4, 5' resulting in '1, 4, 5'. Among these three vertices '1, 4, 5', which one is the first vertex not in (5)? The answer is '1'.

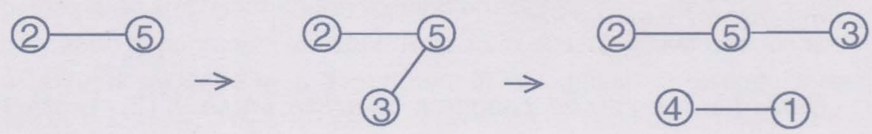
Join '1' to '5' (the first vertex in (5)). Now we have:



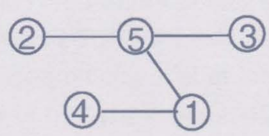
Delete '5' from (5) resulting in ( ). Delete '1' from '1, 4, 5' resulting in '4, 5'. Finally, we join '4' and '5' and arrive at the following spanning tree in A:

Following the procedure described in the first part, it can be checked easily that the above spanning tree of K_5 is indeed associated with the sequence $(4, 1, 5)$.

Take another sequence in B, say $(5, 5, 1)$. Following the same procedure, we obtain the following subgraphs of K_5 step by step:



and arrive at the following spanning tree in A:



Again, the reader can easily check that this spanning tree is indeed associated with the sequence $(5, 5, 1)$ following the procedure described in the first part.

We thus conclude from the above discussion that there is a one-one correspondence between the set A and the set B. Hence,

$$\tau(K_5) = |A| = |B| = 5^3,$$

as was to be shown.

Note. It is interesting to observe that in the sequence $(5,5,1)$, '5' appears twice, '1' appears once, and '2', '3', '4' do not appear at all, whereas in the spanning tree of K_5 associated with $(5,5,1)$, the degree of vertex '5' is $3\ (= 2+1)$, the degree of vertex '1' is $2\ (= 1+1)$, and the degrees of vertices '2', '3' and '4' are all one. As a matter of fact, in general, the degree of a vertex 'j' in a spanning tree T of K_5 is one plus the number of occurrences of 'j' in the sequence associated with T (see Exercise 22.2).

Exercise 22.1 *Let A and B be defined as above for K_5. Explain why different spanning trees in A are associated with different sequences in B.*

Exercise 22.2 *Let T be a spanning tree of K_5 associated with the sequence (s_1, s_2, s_3) as described in the procedure mentioned above. Explain why the degree of a vertex 'j' in T is one plus the number of occurrences of 'j' in (s_1, s_2, s_3).*

Exercise 22.3 *Prove Cayley's formula for general K_n using (i) Matrix-Tree theorem; and (ii) Prufer's one-one correspondence.*

Exercise 22.4 *Consider the special complete bipartite graph $K(2,r)$, where $r \geq 1$. Show that $\tau(K(2,r)) = r2^{r-1}$ by (i) applying the Matrix-Tree theorem; and (ii) any other method.*

Exercise 22.5 *Let $G = K_n - e$, where e is an edge in K_n. Show that*

$$\tau(G) = (n-2)n^{n-3}.$$

References

[1] A Cayley, A theorem on trees, *Quart.J.Pure Appl. Math.* **23**(1889), 376-378 = *Math. Papers*, Vol. 13, 26-28

[2] K. M. Koh, F. M. Dong and E. G. Tay, Graphs and their applications (6), *Mathematical Medley* **32** (1) (2005), 2-10.

[3] J. W. Moon, Various proofs of Cayley's formula for counting trees, A seminar on Graph Theory, Holt, Rinehart and Winston, New York, 1967, 70-78.

[4] H. Prufer, A new proof of a theorem on permutations, *Archiv der Mathematik und Physik* (3) **27** (1918), 142-144 (in German).