# Competition Corner

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Please send your solutions and all other communications about this column to

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**1.** (Italian Mathematical Olympiad, 2005) Let $h$ be a positive integer and let $a_n$ be the sequence defined by the following recursion:

$$a_0 = 1$$
$$a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even} \\ a_n + h & \text{otherwise} \end{cases}$$

(For instance, if $h = 27$, then $a_1 = 28, a_2 = 14, a_3 = 7, a_4 = 34, a_5 = 17, a_6 = 44, \ldots$) For which values of $h$ does there exist $n(\geq 1)$ such that $a_n = 1$?

**2.** (Hong Kong Mathematical Olympiad, 2005) In a school there are $b$ teachers and $c$ students. Suppose that

(i) each teacher teaches exactly $k$ students; and

(ii) for each pair of distinct students, exactly $h$ teaches both of them.

Show that

$$\frac{b}{h} = \frac{c(c-1)}{k(k-1)}.$$

**3.** (Hong Kong Mathematical Olympiad, 2005) On the sides $AB$ and $AC$ of triangle $ABC$, there are points $P$ and $Q$, respectively, such that $\angle APC = \angle AQB = 45°$. Let the perpendicular line to the side $AB$ through $P$ intersects line $BQ$ at $S$. Let the perpendicular line to the side $AC$ through $Q$ intersects line $CP$ at $R$. Let $D$ be on the side $BC$ such that $AD \perp BC$. Prove that the lines $PS, AD, QR$ meet at a common point and that the lines $SR$ and $BC$ are parallel.

**4.** (Swedish Mathematical Olympiad, 2004/2005) Assume that $2n$ (where $n \geq 1$) points are positioned in the plane in such a way that no straight line contains more than two of them. Assume further that $n$ of the points are painted blue, whereas the rest are painted yellow. Show that there exist $n$ segments, each one of them with one blue and one yellow end, such that any of the $2n$ points is an end of a segment and none of the segments intersects another one.

**5.** (19th Nordic Mathematical Contest, 2005) The circle $\mathcal{C}_1$ is inside the circle $\mathcal{C}_2$ and the circles touch each other at $A$. A line through $A$ intersects $\mathcal{C}_1$ again at $B$ and $\mathcal{C}_2$ again at $C$. The tangent to $\mathcal{C}_1$ at $B$ intersects $\mathcal{C}_2$ at $D$ and $E$. The tangents of $\mathcal{C}_1$ passing through $C$ touch $\mathcal{C}_1$ at $F$ and $G$. Prove that $D, E, F$ and $G$ are concyclic.

**6.** (Turkish Mathematical Olympiad, 2005)

(a) For each of the integers $k = 1, 2, 3$, find an integer $n$ such that the number of positive divisors of $n^2 - k$ is 10.

(b) Show that the number of positive divisors of $n^2 - 4$ is not 10 for any value of the integer $n$.

**7.** (Silk Road Mathematical Competition, 2005) Let $A, B, C$ be three collinear points such that $B$ lies in the segment $AC$. Let $AA'$ and $BB'$ be parallel lines such that the points $A'$ and $B'$ lie on the same side of the line $AB$, and $A', B', C$ are not collinear. Let $O_1$ be the centre of the circle passing through the points $A, A', C$ and $O_2$ be the centre of the circle passing through the points $B, B', C$. Find all possible values of $\angle CAA'$ if the areas of $\triangle A'CB'$ and $\triangle O_1CO_2$ are equal.

**8.** (Czech and Slovak Mathematical Olympiad, 2005) An isosceles triangle $KLM$ with base $KL$ is given in the plane. Consider two arbitrary circles $k$ and $l$ which are externally tangent to each other and are tangent to the lines $KM$ and $LM$ at the points $K$ and $L$, respectively. Find the locus of all points of tangency $T$ of such circles $k, l$.

**9.** (Korean Mathematical Olympiad, 2005) Suppose $ABC$ is a triangle with $\angle A = 90°$, $\angle B > \angle A$ and $O$ is the centre of the circumcircle of $\triangle ABC$. Let $l_A$ and $l_B$ be the tangent lines to the circle $O$ at the points $A$ and $B$, respectively, and let $BC$ meet $l_A$ at $S$, $AC$ meet $l_B$ at $D$, $AB$ meet $DS$ at $E$, $CE$ meet $l_A$ at $T$. Also, let $P$ be the point on $l_A$ such that $EP \perp l_A$, $Q(\neq C)$ be the intersection of $CP$ with the circle $O$, $R$ be the intersection of $QT$ with the circle $O$ and $U$ be the interesection of $BR$ with $l_A$. Prove that

$$\frac{SU \cdot SP}{TU \cdot TP} = \frac{SA^2}{TA^2}.$$

**10.** (Korean Mathematical Olympiad, 2005) Find all positive integers $m$ and $n$ such that both $3^m + 1$ and $3^n + 1$ are divisible by $mn$.

## Solutions to the problems of Volume 31 No.2 2004

**1.** (Albanian Mathematical Olympiad, 2000) Prove the inequality

$$\frac{(1+x_1)(1+x_2)\cdots(1+x_n)}{1+x_1x_2\cdots x_n} \le 2^{n-1}, \quad \text{where } x_i \in [1,\infty),\ i=1,\ldots,n$$

When does equality hold?

*Solution by Zhao Yan. Also solved by Kenneth Tay Jingyi, Wu Jiawei, A. Robert Pargeter.*

For any $a,b \in [1,\infty)$, we have $(a-1)(b-1) \ge 0$ or equivalently $2(1+ab) \ge (1+a)(1+b)$, with equality if and only if $a=1$ or $b=1$. Applying this inequality repeatedly, we get

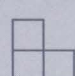
$$\begin{aligned}\frac{(1+x_1)(1+x_2)\cdots(1+x_n)}{1+x_1x_2\cdots x_n} &\le \frac{2(1+x_1x_2)(1+x_3)\cdots(1+x_n)}{1+x_1x_2\cdots x_n} \le \cdots \\ &\le \frac{2^{n-1}(1+x_1x_2\cdots x_n)}{1+x_1x_2\cdots x_n} = 2^{n-1}\end{aligned}$$

with equality if and only if at least one of the elements in each of the pairs

$$(x_1,x_2),\ (x_1x_2,x_3),\ \ldots,\ (x_1x_2\ldots x_{n-1},x_n)$$

is equal to 1. This is certainly satisfied when at most one of the $x_i$'s is $>1$. Moreover, if two of the $x_i$'s, say $x_k$ and $x_l$ with $k<l$, are $>1$, then both elements in the pair $(x_1x_2\ldots x_{l-1},x_l)$ are $>1$. Thus equality holds iff $x_i=1$ for all $i$, with at most one exception.

**2.** (Ukrainian Mathematical Olympiad, 2003) Let $n$ be a positive integer. Some $2n^2+3n+2$ cells of a $(2n+1)\times(2n+1)$ square table are marked. Does there always exist one three-cell figure shown below (such figures can be oriented arbitrarily) such that all three cells are marked?



*Solution by Wu Jiawei. Also solved by Zhao Yan.*

Consider a $2\times(2n+1)$ table $T$. We call the three-cell figure $F$. We note that any $2\times 2$ table will contain $F$ if more than 2 of its cells are marked.

Claim 1: If the marked cells of $T$ do not contain $F$, then $T$ has at most $2n+2$ marked cells. The only way to have $2n+2$ marked cells without $F$ is to marked the cells in the odd numbered columns.

Proof: Divide the first $2n$ columns into $n$ $2\times 2$ subtables and one $1\times 2$ column. If more than $2n+2$ cells are marked, then by the pigeonhole principle, one of the $2\times 2$ subtables will contain 2 cells and thus will contain $F$.

If exactly $2n+2$ cells are marked without having $F$, then each of the $n$ $2 \times 2$ subtables and the $1 \times 2$ column must contain 2 marked cells. For this to be possible, we can only mark the cells in the odd numbered columns.

Claim 2: If $T$ does not contain $F$ and has two adjacent marked cells in one of its rows, then it can contain at most $2n+1$ marked cells.

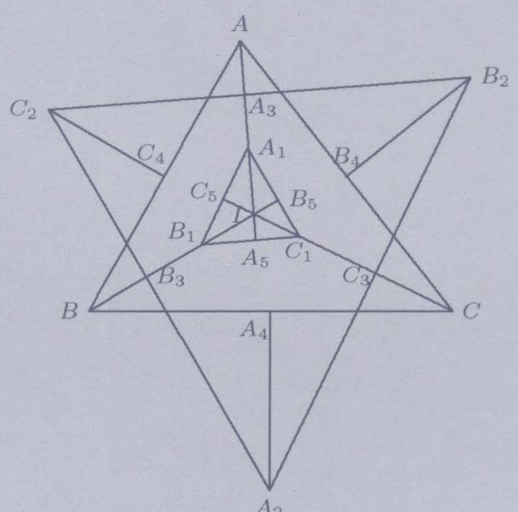
Proof: Suppose the cells in $i$th and $i+1$st position of one of the rows are marked. Assume, without loss of generality, that $i$ is odd. Then divide the columns other than column $i, i+1, i+2$, into $2 \times 2$ subtables. Consider these $n-1$ $2 \times 2$ subtables together with column $i+2$. If there are more than $2n+1$ marked cells, then either column $i+2$ will contain 2 marked cells or one of the $2 \times 2$ subtables will contain 3 marked cells. Therefore $F$ is present, a contradiction.

Now we suppose that there are $2n^2+3n+2$ marked cells in a $(2n+1) \times (2n+1)$ table $S$ and that $F$ is not present. Consider the $n$ $2 \times (2n+1)$ tables formed by the first $2n$ rows together with the last row. By the pigeonhole principle, one of the $2 \times (2n+1)$ tables must contain $2n+2$ marked cells. Since it does not contain $F$, its odd columns are marked by claim 1.

Now consider the $n$ $(2n+1) \times 2$ tables formed by the first $2n$ columns of $T$. By claim 2, each of these tables can contain at most $2n+1$ marked cells. The last column can contain at most $2n+1$ marked cells. Thus the total number of marked cells is $\leq n(2n+1)+(2n+1) = 2n^2+3n+1$, a contradiction. Thus $S$ must contain $F$.

**3.** (Bulgarian Mathematical Olympiad, 2004) Let $I$ be the incentre of $\triangle ABC$ and let $A_1, B_1, C_1$ be arbitrary points on the segments $AI, BI, CI$, respectively. The perpendicular bisectors of $AA_1$, $BB_1$, $CC_1$ intersect at $A_2, B_2, C_2$. Prove that the circumcentre of $A_2B_2C_2$ coincides with the circumcentre of $ABC$ if and only if $I$ is the orthocentre of $A_1B_1C_1$.

*Solution by Zhao Yan.* Let $\angle BAI = \angle CAI = \alpha$, $\angle ABI = \angle CBI = \beta$, $\angle ACI = \angle BCI = \gamma$. Then $\alpha + \beta + \gamma = 90°$.



Let $A_2B_2$ intersect $CI$ at $C_3$, $B_2C_2$ intersect $AI$ at $A_3$, $C_2A_2$ intersect $BI$ at $B_3$, $A_1B_1$ intersect $CI$ at $C_5$, $B_1C_1$ intersect $AI$ at $A_5$, $C_1A_1$ intersect $BI$ at $B_5$. Let $A_4, B_4, C_4$ be points on $BC, CA, AB$, respectively such that $A_2A_4 \perp BC$, $B_2B_4 \perp AC$, $C_2C_4 \perp AB$.

Since $\angle AA_3C_2 = \angle AC_4C_2 = 90°$, $A, A_3, C_4, C_2$ are concyclic. Similarly $C_2, C_4, B_3, B$ are concyclic. Therefore $\angle C_4C_2A_3 = \angle C_4AA_3 = \alpha$, $\angle C_4C_2B_3 = \angle C_4BB_3 = \beta$. so $\angle B_2C_2A_2 = \alpha + \beta$. Similarly, $\angle A_2B_2C_2 = \alpha + \gamma$, $\angle C_2A_2B_2 = \beta + \gamma$.

Let $A_2A_4$ and $B_2B_4$ intersect at $O$. Since $OB_2A_2 = \angle OA_2B_2 = \gamma$. Therefore, $OA_2 = OB_2$, $\angle A_2OB_2 = 180° - 2\gamma = 2(\alpha + \beta) = 2\angle A_2C_2B_2$. Consequnetly, $O$ is the circumcentre of $\triangle A_2B_2C_2$. Similar consideration will show that $C_2C_4$ passes through $O$ as well.

If $I$ is the orthocentre of $\triangle A_1B_1C_1$, then $A_1A_5 \perp B_1C_1$, $B_1B_5 \perp A_1C_1$, $CA_1C_5 \perp A_1B_1$ and

$$\angle IC_1B_1 = 90° - \angle C_1IA_5 = 90° - \angle IAC - \angle ICA = 90° - \alpha - \gamma = \beta = \angle B_1BC.$$

Therefore $B_1, C_1, B, C$ are concyclic.

Since $A_2C_2$, $A_2B_2$ are perpendicular bisectors of $BB_1$ and $CC_1$, respectively, they meet at the circumcentre of $B_1C_1CB$. Therefore $A_2$ is the circumcentre of $B_1C_1CB$. Since $BC \perp A_2A_4$, $A_4$ bisects $BC$. Similarly, $B_4$ bisects $AC$ and $C_4$ bisects $AB$. Therefore $O$ is the circumcentre of $\triangle ABC$. Thus $\triangle ABC$ and $\triangle A_2B_2C_2$ have the same circumcentre $O$.

Now suppose that $\triangle ABC$ and $\triangle A_2B_2C_2$ have the same circumcentre $O$. Then $A_4O, B_4O, C_4O$ are perpendicular bisectors of $BC, CA, AB$.

In $\triangle BC_1C$, $A_2C_3$ and $A_2A_4$ are perpendicular bisectors of $C_1C$ and $BC$, respectively. Therefore $A_2$ is the circumcentre of $\triangle BC_1C$. Similarly, $A_2$ is the circumcentre of $\triangle BB_1C$. Therefore $B, B_1, C_1, C$ are concyclic. Thus $\angle IC_1B_1 = \angle IBC = \beta$. Since $\angle A_5IC_1 = \angle IAC + \angle ICA = \alpha + \gamma$, we have $\angle IA_5C_1 = 180° - \angle IC_1B_1 - \angle A_5IC_1 = 90°$. Therefore $A_1A_5 \perp B_1C_1$. Similarly, $B_1B_5 \perp A_1C_1$, $C_1C_5 \perp B_1A_1$. Therefore $I$ is the orthocentre of $\triangle A_1B_1C_1$.

**4.** (Russian Mathematical Olympiad, 2004) The distance between two 5-digit numbers $\overline{a_1a_2a_3a_4a_5}$ and $\overline{b_1b_2b_3b_4b_5}$ is the maximal integer $i$ for which $a_i \neq b_i$. All the 5-digit numbers are written down one by one in some order. What is the minimal possible sum of distances between adjacent numbers?

*Solution by Zhao Yan.* Let $L$ be a list of all 5-integers. For each $i = 1, 2, 3, 4, 5$, let $x_i$ be the number of pairs with distance $i$. We want to find the minimum value of

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5.$$

Note that $x_1 + x_2 + x_3 + x_4 + x_5 = 89999$.

Claim: For each $i = 2, 3, 4, 5$, $x_i + \cdots + x_5 \geq 10^{6-i} - 1$.

Proof: Consider the set $S$ of all $(6-i)$-digit numbers, where the first digit can be 0. Then $|S| = 10^{6-i}$. For each element $x \in S$, let $n$ be the first element in $L$ whose final $6-i$ digits are $x$. If $n$ is not the first element of $L$, then the final $6-i$ digits of the number, say $m$, preceding it in the list $L$ are not $x$. Thus the distance between $m$ and $n$ is at least $i$. So the number of adjacent pairs with distance at least $i$ is $\geq |S| - 1 = 10^{6-i} - 1$. This proves the claim.

Thus we have

$$\begin{aligned} x_2 + x_3 + x_4 + x_5 &\geq 9999 \\ x_3 + x_4 + x_5 &\geq 999 \\ x_4 + x_5 &\geq 99 \\ x_5 &\geq 9 \end{aligned}$$

Therefore

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 \geq 89999 + 9999 + 999 + 99 + 9 = 101105.$$

Next the following list of 5-digit integers shows that this bound can be achieved, showing that the answer is 101105. The first number in the list is 10000. If $a_1a_2a_3a_4a_5$ is a number in the list and $i$ is the smallest index such that $a_i \neq 9$, then the next number is obtained by replacing $a_i$ by $a_i + 1$, $a_1$ by 1 and $a_2, \ldots, a_{i-1}$ by 0. For example, the number 99139 is followed by 10239 and 21111 is floowed by 31111. It can be checked easily that $x_1 = 8000$, $x_2 = 9000$, $x_3 = 900$, $x_4 = 90$ and $x_5 = 9$.

The solver also noted that the answer would be 111105 if the first digit were allowed to be 0.

**5.** (Hungarian Mathematical Olympiad, 2002/3) Let $n$ be an integer, $n \geq 2$. We denote by $a_n$ the greatest number with $n$ digits which is neither the sum nor the difference of two perfect squares. (a) Determine $a_n$ as a function of $n$. (b) Find the smallest value of $n$ for which the sum of squares of the digits of $a_n$ is a perfect square.

*Solution by Zhao Yan and Kenneth Tay Jingyi.*

Note that $k^2 \equiv 0, 1, 4 \pmod 8$. Thus $k^2 - \ell^2 \equiv 0, \pm1, \pm3, 4 \pmod 8$ and $k^2+\ell^2 \equiv 0, 1, 2, 4, 5 \pmod 8$. Thus any integer $m \equiv 6 \pmod 8$ cannot be expressed as the sum or difference of 2 perfect squares. (a) We claim that

$$a_n = \begin{cases} 10^n - 2 & \text{if } n > 2 \\ 94 & \text{if } n = 2 \end{cases}$$

Case (i) $n > 2$:

$$\begin{aligned} 10^n - 2 &\equiv 10^{n-3}10^3 - 2 \\ &\equiv 6 \pmod 8 \end{aligned}$$

Also $10^n - 1 = (5 \cdot 10^{n-1})^2 - (5 \cdot 10^{n-1} - 1)^2$.

Case (ii) $n = 2$: $94 \equiv 6 \pmod 8$ and $95 = 48^2 - 47^2$, $96 = 25^2 - 23^2$, $97 = 49^2 - 48^2$, $98 = 7^2 + 7^2$, $99 = 10^2 - 1^2$.

Thus the claim is established.

(b) For $n = 2$, $9^2 + 4^2 = 97$ is not a perfect square. For $n > 2$, the sum of the squares of the digits of $a_n$ is $(n-1)9^2 + 8^2 = 81n - 17$. Thus we want to the smallest $n$ such that $81n - 17 = k^2$. We have $k^2 \equiv 64 \pmod{81}$. Thus $(k-8)(k+8) \equiv 0 \pmod{81}$. Since $\gcd(k-8, k+8) = \gcd(k-8, 16)$, we have $81 \mid k+8$ or $81 \mid k-8$. Thus $k = 81m \pm 8$. The smallest possible $k = 81 - 8 = 73$. (We know $k \neq 8$.) This gives $n = 66$ as the smallest value of $n$.

**6.** (Thai Mathematical Olympiad, 2003) Find all primes $p$ such that $p^2 + 2543$ has less than 16 distinct positive divisors.

*Solution by Zhao Yan.* When $p = 2$ and $p = 3$, $p^2 + 2543$ has 6 and 16 factors, respectively. For $p > 3$, $p^2 \equiv 1 \pmod{24}$ since $p^2 \equiv 1 \pmod 3$ and $p^2 \equiv 1 \pmod 8$. Thus $p^2 + 2543 \equiv 0 \pmod{24}$.

Case (i) $p^2 + 2543 = 2^\alpha 3^\beta$: We have

$$2543 < p^2 + 2543 = 2^\alpha 3^\beta < 3^{\alpha+\beta}.$$

But $3^7 < 2543 < 3^8$. Thus $\alpha + \beta \geq 8$. Since $24 \mid p^2 + 2543$, $\alpha \geq 3$, $\beta \geq 1$. Therefore $(\alpha - 1)(\beta - 1) \geq 0$, i.e., $\alpha\beta \geq \alpha + \beta - 1$. Hence $(\alpha+1)(\beta+1) = \alpha\beta + \alpha + \beta + 1 \geq 2(\alpha + \beta) \geq 16$. Thus there are at least 16 factors.

Case (ii) $p^2 + 2543 = 2^\alpha 3^\beta A$ for some $A$ such that $3 \nmid A$ and $2 \nmid A$: As before, we have $\alpha \geq 3$ and $\beta \geq 1$. Thus the number of factors is $\geq 4 \times 2 \times 2 = 16$.

Thus the only such prime is $p = 2$.

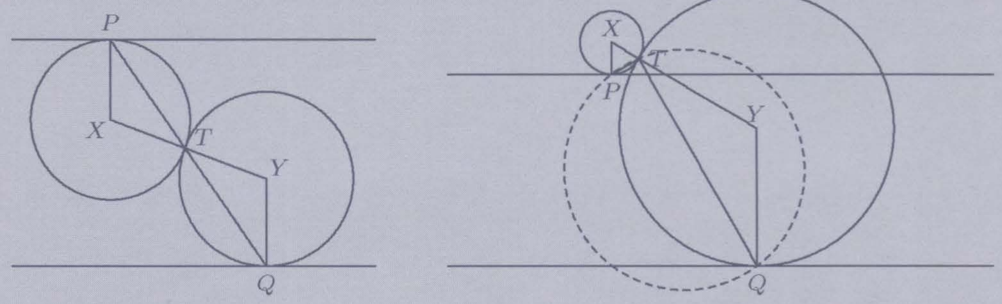
**7.** (Italian Mathematical Olympiad, 2003/4) Let $r$ and $s$ be two parallel lines and $P, Q$ be points on $r$ and $s$, respectively. Consider the pair $(C_P, C_Q)$ where $C_P$ is a circle tangent to $r$ at $P$, $C_Q$ is a circle tangent to $s$ at $Q$ and $C_P, C_Q$ are tangent externally to each other at some point, sat $T$. Find the locus of $T$ when $(C_P, C_Q)$ varies over all pairs of circles with the given properties.

*We present the combined solution of A. Robert Pargeter, Zhao Yan and Wu Jiawei.*

Case (i): both circles lie between $r, s$. Iftheir centres are $X$ and $Y$, then $XTY$ is a straight line and it's easy to see that $\triangle PXT \simeq \triangle QYT$, so that $PTQ$ is a straight line.(see figure).

Conversely, let $T$ be a point on the segment $PQ$. Let $X$ be the point such that $XP \perp r$ and $XP = XT$ and $Y$ be the point such that $YQ \perp s$ and $YQ = YT$. Then it's again easy to see that $\triangle PXT \simeq \triangle QYT$, so that $X, T, Y$ are collinear. Hence the circles with centres $X$ and $Y$ and radius $XP$ and $YQ$ are tangent externally at $T$.

Thus the locus of $T$ is the segment $PQ$.




Case (ii): $C_P$ lies above $r$ and $C_Q$ above $s$. Let $C$ be the circle with diameter $PQ$. We claim that the locus of $T$ is the part $C$ above the line $r$.

Since $QY \parallel PX$ and $XTY$ is a striaght line, we have $\angle PXT + \angle QYT = 180°$. From this it's easy to see that $\angle PTQ = 90°$. Thus $T$ is on $C$.

Conversely, let $T$ be a point on the part of $C$ above $r$. Construct the points $X, Y$ such that $YQ \perp s$, $XP \perp r$, $YT = YQ$ and $XP = XT$. Since $\angle PTQ = 90°$ and $QY \parallel PX$, we have $\angle YQT + \angle XPT = 90°$. This in turn implies that $\angle QTY + \angle PTX = 90°$. Thus $X, T, Y$ are collinear. Hence $T$ is the point of common tangency of the circles, one with centre $X$, radius $XP$ and the other with centre $Y$, radius $YQ$.

The other case is symmetrical with (ii). Thus the locus is the segment $PQ$ and those parts of the circle on $PQ$ as diameter outside the parallels.

Note: A. Robert Pargeteralso points out that if we allow the circles to touch internally, then the locus is the line $PQ$ produced and the whole of the circle $C$.

**8.** (Estonian Mathematical Olympiad, 2003/4) (a) Does there exist a convex quadrilateral $ABCD$ satisfying the following conditions:

(1) $ABCD$ is not cyclic;

(2) the sides $AB$, $BC$, $CD$ and $DA$ have pairwise different lengths;

(3) the circumradii of the triangles $ABC$, $BAD$ and $BCD$ are equal?*

(b) Does there exist such a non-convex quadrilateral?

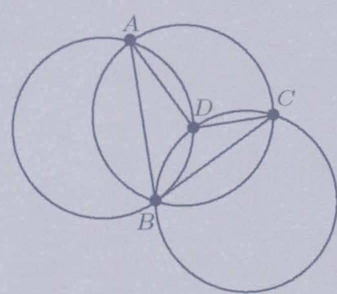
*Solution by Wu Jiawei.*

(a) Suppose such a quadrilateral exists. We first note that for any $\triangle XYZ$, its circumradius $R$ is given by $(XY \sin Z)/2$. Since the circumradii of triangles $BAD$ and $BCD$ are equal, we have $BD \sin BAD = BD \sin BCD$ and thus $\sin BAD = D \sin BCD$. But $ABCD$ is not cyclic. Therefore $\angle BAD = \angle BCD = 180°$. Since the circumradii of triangles $ABC$ and $BAD$ are equal, we have $BA \sin BCA = BA \sin BDA$ and thus $\sin BCA = D \sin BDA$. But $\angle BCA \neq \angle BDA$. Thus $\angle BCA + \angle BDA = 180°$. Now in $\triangle ABD$, we have

$$\angle BAD + \angle BDA = \angle BCD + \angle BDA > \angle BCA + \angle BDA = 180°$$

a contradiction ($\angle BCD > \angle BCA$ by the convexity of $ABCD$.)

(b) Such a non-convex quadrilateral exists. Draw three equal circles which meet at a common point $B$. Let $A, C, D$ be points common to two of the circles. Then $ABCD$ is the desired quadrilateral.




**9.** (Indian Mathematical Olympiad, 2004) Let $S$ denote the set of all 6-tuples $(a, b, c, d, e, f)$ of positive integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Consider the set

$$T = \{abcdef : (a, b, c, d, e, f) \in S\}.$$

Find the greatest common divisor of all the members of $T$.

*Solution by Zhao Yan. Also solved by Kenneth Tay Jingyi.*

Let $d$ be the required greatest common divisor. Note that $(1, 1, 1, 2, 3, 4) \in S$. Thus $24 \in T$. Hence $d \mid 24$.

Let $(a, b, c, d, e, f) \in S$. if $3 \nmid abcdef$, then $a^2 + \cdots + e^2 \equiv 5 \equiv 2 \equiv f^2 \pmod 3$, a contradiction. Thus $3 \mid abcdef$.

Let $2^m$ be the highest power of 2 that divides $abcde$. First note that, modulo 8,

$$n^2 \equiv \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0, 4 & \text{if } n \text{ is even} \end{cases}$$

If $m = 0$, then $a^2 + \cdots + e^2 \equiv 5 \equiv f^2 \pmod 8$, a contradiction. Thus $m \neq 0$.

If $m = 1$, then $a^2 + \cdots + e^2 \equiv 0 \equiv f^2 \pmod 8$. Thus $4 \mid f$ and $8 \mid abcdef$.

If $m = 2$, and exactly one of $a, b, c, d, e$ is even, then $f$ is even as well and so $8 \mid abcdef$.

If $m = 2$ and exactly two of $a, b, c, d, e$ is even, then $a^2 + \cdots + e^2 \equiv 3 \equiv f^2 \pmod 8$, a contradiction.

So in all cases, we have $8 \mid abcdef$. So $d = 24$.

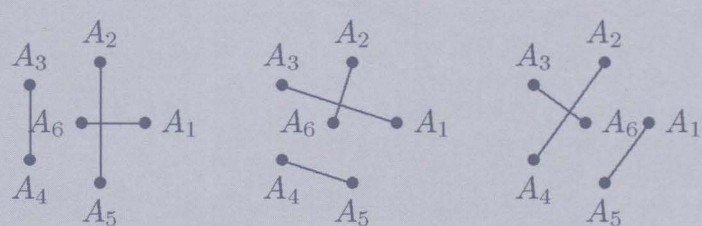
**10.** (Ukrainian Mathematical Olympiad, 2004) Let $A_1, A_2, \ldots, A_{2004}$ be the vertices of a convex 2004-gon(i.e., a polygon with 2004 sides). Is it possible to mark each side and each diagonal of the polygon with one of 2003 colours in such a way that the following two conditions hold:

(1) there are 1002 segments of each colour;

(2) if an arbitrary vertex and two arbitrary colours are given, one can start from this vertex and, using segments of these two colours exclusively, visit every other vertex only once?

*Solution by the editor.* The actual location of the vertices are not important. The problem is about colouring the edges of the complete graph on $n$ vertices, $n$ even. We shall place vertices $A_1, \ldots, A_{n-1}$ evenly on the circumference of a circle and $A_n$ at the centre.

Colour the edge $A_nA_i$ and all the edges perpendicular to it with colour $i$. This certainly satisfies condition (1). To satisfy condition (2), we need to show that for any two colours $i, j$, the edges with these two colours form a cycle. (The figure below shows for $n = 6$, the edges with colours 1, 2, 3. It's easy to see that any two of the three sets of edges form a cycle.




Without loss of generality, we only need to show this for the case of colours 1 and $i$. In the subgraph formed by these two sets of edges, every vertex is of degree 2. Thus it is the union of edge disjoint cycles. To show that it is a single cycle, we only need to show that the subgraph is connected. For any vertex $A_a$, the edge $A_aA_{2j-a}$, (here the subscripts are taken mod $n-1$), is perpendicular to $A_nA_j$. Thus it has colour $j$. Thus the edge $A_aA_{2-a}$ is of colour 1 and the edge $A_{2-a}A_{2i-2+a}$ is of colour $i$. Thus $A_a$ is connected to $A_{2(i-1)+a}$. Hence $A_1$ is connected to $A_{2p(i-1)+1}$, $p = 1, \ldots, n-2$. But these vertices are distinct since $2j+1 \equiv 2k+1 \pmod{n-1}$ implies that $2(j-k) \equiv 0 \pmod{n-1}$. But $n-1$ is odd. Therefore $j = k$ if they are both $\leq n-1$. Hence the subgraph is connected are required.