Pascal's Triangle and the General Binomial Theorem

by
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It is a well-known fact that the coefficients of terms in the expansion of $(1+x)^n$ for non-negative integer n can be arranged in the following manner known as the Pascal's Triangle:

$$
\begin{array}{lc}
n=0 & 1 \\
n=1 & 1 \quad 1 \\
n=2 & 1 \quad 2 \quad 1 \\
n=3 & 1 \quad 3 \quad 3 \quad 1 \\
n=4 & 1 \quad 4 \quad 6 \quad 4 \quad 1 \\
\ldots\ldots & \ldots\ldots\ldots\ldots
\end{array}
$$

By elementary mathematics, we have

$$(1+x)^n = C_0^m + C_1^m x + C_2^m x^2 + \cdots + C_{n-1}^m x^{n-1} + C_n^m x^n,$$

where

$$C_r^m = \frac{n!}{r!(n-r)!} \text{ for non-negative integer } n \text{ and } r = 0, 1, 2, \ldots, n \tag{1}$$

This is the Binomial Theorem with non-negative integer as index. Thus, a Pascal's Triangle can be written as

$$
\begin{array}{c}
C_0^0 \\
C_0^1 \quad C_1^1 \\
C_0^2 \quad C_1^2 \quad C_2^2 \\
C_0^3 \quad C_1^3 \quad C_2^3 \quad C_3^3 \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{array}
$$

By tracking the terms in the expansion of $(1+x)^m(1+x)^n$, we obtain the result

$$C_r^{m+n} = C_r^m C_0^n + C_{r-1}^m C_1^n + C_{r-2}^m C_2^n + \cdots + C_1^m C_{r-1}^n + C_0^m C_r^n \tag{2}$$

for non-negative integers m, n and r where $C_s^t = 0$ if $t < s$.

The generalisation of the Binomial Theorem to real index is quite complex. We shall see how the Pascal's Triangle can help. For brevity of expression, we define

$$\alpha^{(0)} = 1,\ \alpha^{(1)} = \alpha,\ \alpha^{(2)} = \alpha(\alpha-1),\ \alpha^{(3)} = \alpha(\alpha-1)(\alpha-2), \cdots\cdots$$

for any real number α.

We take note of the difference between $\alpha^{(n)}$ and α^n, where n is a non-negative integer. We may describe $\alpha^{(n)}$ as step-down n-product of α, whereas α^n is a self-n-product.

Result (2) can be written as

$$\frac{(m+n)^{(r)}}{r!} = \frac{m^{(r)}}{r!}\frac{n^{(0)}}{0!} + \frac{m^{(r-1)}}{(r-1)!}\frac{n^{(1)}}{1!} + \frac{m^{(r-2)}}{(r-2)!}\frac{n^{(2)}}{2!} + \cdots + \frac{m^{(1)}}{1!}\frac{n^{(r-1)}}{(r-1)!} + \frac{m^{(0)}}{0!}\frac{n^{(r)}}{r!},$$

for non-negative integers m and n and $r = 0, 1, 2, \ldots$ (3)

We shall proceed to show that result (3) still hold when m and n are replaced by any real numbers.

Consider expansions of $(\alpha+\beta)^{(r)}$ in terms of $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)} \ldots$ and $\beta^{(0)}, \beta^{(1)}, \beta^{(2)}, \ldots$ for all real numbers α and β and $r = 0, 1, 2, \ldots$. We have

$$\begin{aligned}
(\alpha+\beta)^{(0)} &= 1. \\
(\alpha+\beta)^{(1)} &= \alpha+\beta = \alpha^{(1)} + \beta^{(1)}. \\
(\alpha+\beta)^{(2)} &= (\alpha+\beta)(\alpha+\beta-1) = \alpha(\alpha-1) + \alpha\beta + \beta\alpha + \beta(\beta-1) \\
&= \alpha(\alpha-1) + 2\alpha\beta + \beta(\beta-1) = \alpha^{(2)} + 2\alpha^{(1)}\beta^{(1)} + \beta^{(2)}. \\
(\alpha+\beta)^{(3)} &= (\alpha+\beta)(\alpha+\beta-1)(\alpha+\beta-2) = [\alpha^{(2)} + 2\alpha^{(1)}\beta^{(1)} + \beta^{(2)}](\alpha+\beta-2) \\
&= \alpha^{(2)}(\alpha-2) + \alpha^{(2)}\beta + 2\alpha^{(1)}\beta^{(1)}(\alpha-1) + 2\alpha^{(1)}\beta^{(1)}(\beta-1) + \beta^{(2)}\alpha + \beta^{(2)}(\beta-2) \\
&= \alpha^{(3)} + \alpha^{(2)}\beta^{(1)} + 2\alpha^{(2)}\beta^{(1)} + 2\alpha^{(1)}\beta^{(2)} + \alpha^{(1)}\beta^{(2)} + \beta^{(3)} \\
&= \alpha^{(3)} + 3\alpha^{(2)}\beta^{(1)} + 3\alpha^{(1)}\beta^{(2)} + \beta^{(3)} = \cdots\cdots
\end{aligned}$$

We notice that the results are analogous to those for $(\alpha+\beta)^0$, $(\alpha+\beta)^1$, $(\alpha+\beta)^2$, $(\alpha+\beta)^3$, $\cdots$ in terms of $\alpha^0, \alpha^1, \alpha^2, \ldots$ and $\beta^0, \beta^1, \beta^2, \ldots$ for all real numbers α and β.

It is a conjecture now that in the expansions of $(\alpha+\beta)^{(r)}$ in terms of $\alpha^{(0)}, \alpha^{(1)}, \alpha^{(2)}, \ldots$ and $\beta^{(0)}, \beta^{(1)}, \beta^{(2)}, \ldots$ for $r = 0, 1, 2, \ldots$, the coefficients can be arrayed in Pascal's Triangle.

So, intuitively, we conjecture the Key Result (KS) as follows:

$$(\alpha+\beta)^{(r)} = C_0^r\alpha^{(r)} + C_1^r\alpha^{(r-1)}\beta^{(1)} + C_2^r\alpha^{(r-2)}\beta^{(2)} + \cdots + C_{r-1}^r\alpha^{(1)}\beta^{(r-1)} + C_r^r\beta^{(r)},$$

analogous to

$$(\alpha+\beta)^r = C_0^r\alpha^r + C_1^r\alpha^{r-1}\beta^1 + C_2^r\alpha^{r-2}\beta^2 + \cdots + C_{r-1}^r\alpha^1\beta^{(r-1)} + C_r^r\beta^r.$$

We shall use mathematical induction for a rigorous proof of the KS. It is clear that the statement is true for $r = 0, 1$. Further, if it is true for $r = k$, then

$$(\alpha+\beta)^{(k)} = \sum_{i=0}^{k} C_i^k \alpha^{(k-i)}\beta^{(i)}.$$

We shall obtain the expansion for $(\alpha+\beta)^{(k+1)}$ from $(\alpha+\beta)^{(k)}$, just as we obtained the expansion of $(\alpha+\beta)^{(3)}$ from $(\alpha+\beta)^{(2)}$.

$$\begin{aligned}
(\alpha+\beta)^{(k+1)} &= (\alpha+\beta)^{(k)}(\alpha+\beta-k) = (\alpha+\beta)^{(k)}[(\alpha-k+i)+(\beta-i)] \\
&= \sum_{i=0}^{k} C_i^k\alpha^{(k-i)}\beta^{(i)}(\alpha-k+i) + \sum_{i=0}^{k} C_i^k\alpha^{(k-i)}\beta^{(i)}(\beta-i) \\
&= \sum_{i=0}^{k} C_i^k\alpha^{(k-i+1)}\beta^{(i)} + \sum_{i=0}^{k} C_i^k\alpha^{(k-i)}\beta^{(i+1)} \\
&= C_0^k\alpha^{(k+1)}\beta^{(0)} + \sum_{i=1}^{k} C_i^k\alpha^{(k-i+1)}\beta^{(i)} + \sum_{i=0}^{k-1} C_i^k\alpha^{(k-i)}\beta^{(i+1)} + C_k^k\alpha^{(0)}\beta^{(k+1)} \\
&= C_0^k\alpha^{(k+1)}\beta^{(0)} + \sum_{i=1}^{k} C_i^k\alpha^{(k-i+1)}\beta^{(i)} + \sum_{i=1}^{k} C_{i-1}^k\alpha^{(k-i+1)}\beta^{(i)} + C_k^k\alpha^{(0)}\beta^{(k+1)} \\
&= C_0^k\alpha^{(k+1)}\beta^{(0)} + \sum_{i=1}^{k} C_i^{k+1}\alpha^{(k+1-i)}\beta^{(i)} + C_k^k\alpha^{(0)}\beta^{(k+1)}, \text{ using } C_i^k + C_{i-1}^k = C_i^{k+1}, \\
&= \sum_{i=0}^{k+1} C_i^{k+1}\alpha^{(k+1-i)}\beta^{(i)}, \text{ using } C_0^k = C_0^{k+1} \text{ and } C_k^k = C_{k+1}^{k+1}.
\end{aligned}$$

Hence we complete the proof of the KS. It follows that:

$$\frac{(\alpha+\beta)^{(r)}}{r!} = \frac{\alpha^{(r)}}{r!}\frac{\beta^{(0)}}{0!} + \frac{\alpha^{(r-1)}}{(r-1)!}\frac{\beta^{(1)}}{1!} + \frac{\alpha^{(r-2)}}{(r-2)!}\frac{\beta^{(2)}}{2!} + \cdots + \frac{\alpha^{(0)}}{0!}\frac{\beta^{(r)}}{r!}$$

is valid for all real numbers α, β and $r = 0, 1, 2, \ldots$ (4)

We shall see how this leads to the general Binomial Theorem:

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha^{(2)}}{2!}x^2 + \frac{\alpha^{(3)}}{3!}x^3 + \cdots$$

where $\alpha^{(r)} = \alpha(\alpha-1)(\alpha-2)\ldots(\alpha-r+1)$, for all real number α, provided that the infinite series in the case of α not being a non-negative integer converges (later we will show that a sufficient requirement for this is $|x| < 1$). (5)

Consider the product of two convergent series:

$$(1+\alpha x+\frac{\alpha^{(2)}}{2!}x^2+\frac{\alpha^{(3)}}{3!}x^3+\cdots)(1+\beta x+\frac{\beta^{(2)}}{2!}x^2+\frac{\beta^{(3)}}{3!}x^3+\cdots),$$

the term x^r is

$$(\frac{\alpha^{(r)}}{r!}\frac{\beta^{(0)}}{0!}+\frac{\alpha^{(r-1)}}{(r-1)!}\frac{\beta^{(1)}}{1!}+\frac{\alpha^{(r-2)}}{(r-2)!}\frac{\beta^{(2)}}{2!}+\cdots+\frac{\alpha^{(1)}}{1!}\frac{\beta^{(r-1)}}{(r-1)!}+\frac{\alpha^{(0)}}{0!}\frac{\beta^{(r)}}{r!})x^r$$

which is $\frac{(\alpha+\beta)^{(r)}}{r!}x^r$, by result (4). Therefore,

$$(1+\alpha x+\frac{\alpha^{(2)}}{2!}x^2+\frac{\alpha^{(3)}}{3!}x^3+\cdots)(1+\beta x+\frac{\beta^{(2)}}{2!}x^2+\frac{\beta^{(3)}}{3!}x^3+\cdots)$$
$$=1+(\alpha+\beta)x+\frac{(\alpha+\beta)^{(2)}}{2!}x^2+\frac{(\alpha+\beta)^{(3)}}{3!}x^3+\cdots,$$

the product of two convergent series, is again a convergent series. (6)

Now, for $m, n \in \mathbb{Z}^+$, we have, by result (6),

$$[1+\frac{(\frac{m}{n})}{1}x+\frac{(\frac{m}{n})^{(2)}}{2!}x^2+\frac{(\frac{m}{n})^{(3)}}{3!}x^3+\cdots]^n=1+\frac{(m)}{1}x+\frac{(m)^{(2)}}{2!}x^2+\frac{(m)^{(3)}}{3!}x^3+\cdots=(1+x)^m,$$

since the sum of n terms, $\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\cdots+\frac{m}{n}$ is m. Thus,

$$1+\frac{(\frac{m}{n})}{1}x+\frac{(\frac{m}{n})^{(2)}}{2!}x^2+\frac{(\frac{m}{n})^{(3)}}{3!}x^3+\cdots=(1+x)^{\frac{m}{n}}.$$

Again, we see that if

$$1+\alpha x+\frac{\alpha^{(2)}}{2!}x^2+\frac{\alpha^{(3)}}{3!}x^3+\cdots=(1+x)^\alpha,$$

then by result (6),

$$[1+\alpha x+\frac{\alpha^{(2)}}{2!}x^2+\frac{\alpha^{(3)}}{3!}x^3+\cdots][1+(-\alpha)x+\frac{(-\alpha)^{(2)}}{2!}x^2+\frac{(-\alpha)^{(3)}}{3!}x^3+\cdots]=1,$$

this implies that

$$1+(-\alpha)x+\frac{(-\alpha)^{(2)}}{2!}x^2+\frac{(-\alpha)^{(3)}}{3!}x^3+\cdots=[(1+x)^\alpha]^{-1}=(1+x)^{-\alpha}.$$

We can thus conclude that

$$(1+x)^p=1+px+\frac{p^{(2)}}{2!}x^2+\frac{p^{(3)}}{3!}x^3+\cdots,$$

for any rational number p (positive or negative), provided that the infinite series converges. If p is not a non-negative integer, the series may not converge. (7)

We now consider $(1+x)^\alpha$ for any irrational number α.

Let $\{p\}$ be a sequence of rational numbers $p_1, p_2, p_3, \ldots$ such that $\lim_{i\to\infty} p_i = \alpha$. We have $(1+x)^\alpha = \lim_{i\to\infty}(1+x)^{p_i}$.

For brevity, we write, for any real number β and any positive integer r,

$$\begin{aligned} A(x,\beta,r) \quad &\text{for} \quad 1+\frac{\beta}{1!}x+\frac{\beta^{(2)}}{2!}x^2+\cdots+\frac{\beta^{(r)}}{r!}x^r, \\ B(x,\beta,r) \quad &\text{for} \quad \frac{\beta^{(r+1)}}{(r+1)!}x^{r+1}+\frac{\beta^{(r+2)}}{(r+2)!}x^{r+2}+\cdots, \end{aligned}$$

and

$$A(x,\beta,\infty) \quad \text{for} \quad 1+\frac{\beta}{1!}x+\frac{\beta^{(2)}}{2!}x^2+\cdots.$$

Clearly, $A(x,\beta,\infty) = A(x,\beta,r)+B(x,\beta,r)$. Also, let the term $\frac{\beta^{(r)}}{r!}x^r$ be denoted by $u_r(x,\beta)$. Then $\frac{u_{r+1}(x,\beta)}{u_r(x,\beta)} = \frac{\beta-r}{r+1}x = \nu_r(x,\beta)$.

Given $|x|<1$, and any small positive value ε, we can find δ and choose $\{p\}$ such that $|x| < 1-\delta < 1$ and $p_i \in [\alpha-\delta, \alpha+\delta]$ for all i with $\lim_{i\to\infty} p_i = \alpha$.

For any real number β such that

$$\alpha-\delta \le \beta \le \alpha+\delta,$$

$$\nu_r(x,\beta) = \frac{\beta-r}{r+1}x = (-1+\frac{\beta+1}{r+1})x,$$

$$-1+\frac{\alpha+1-\delta}{r+1} \le \frac{\beta-r}{r+1} \le -1+\frac{\alpha+1+\delta}{r+1} < 0 \quad \text{for large } r,$$

$$\left|\frac{\beta-r}{r+1}\right| \le \left|-1+\frac{\alpha+1-\delta}{r+1}\right| \quad \text{for large } r, \text{ and } \lim_{r\to\infty}\left|-1+\frac{\alpha+1-\delta}{r+1}\right| = 1.$$

There exists r_1, independent of β, such that for $r \ge r_1$,

$$|\nu_r(x,\beta)| = \left|\frac{\beta-r}{r+1}x\right| < 1-\delta, \quad \text{since } |x| < 1-\delta < 1.$$

Noting that $\frac{\beta^{(r_1)}}{r_1}x^{r_1}$ is a continuous function of $\beta \in [\alpha-\delta, \alpha+\delta]$, for fixed value of x and r_1, let M be an upper bound of $\left|\frac{\beta^{(r_1)}}{r_1}x^{r_1}\right|$. We have, because $\left|\frac{\beta^{(r_1)}}{(r_1)!}x^{r_1}\right| \le M$,

$$\begin{aligned} \left|\frac{\beta^{(r_1+1)}}{(r_1+1)!}x^{r_1+1}\right| &\le M|\nu_{r_1}(x,\beta)| < M(1-\delta), \\ \left|\frac{\beta^{(r_1+2)}}{(r_1+2)!}x^{r_1+2}\right| &\le M|\nu_{r_1}(x,\beta)\nu_{r_1+1}(x,\beta)| < M(1-\delta)^2, \end{aligned}$$

$$\cdots\cdots$$

There exists s such that

$$\left|\frac{\beta^{(r_1+s)}}{(r_1+s)!}x^{r_1+s}\right| < M(1-\delta)^s < \varepsilon\delta.$$

In short, there exists r_2, independent of β , such that for $r > r_2$, $\left|\frac{\beta^{(r)}}{(r)!}x^r\right| < \varepsilon\delta$. Now,

$$\begin{aligned} B(x,\beta,r) &= \frac{\beta^{(r+1)}}{(r+1)!}x^{r+1} + \frac{\beta^{(r+2)}}{(r+2)!}x^{r+2} + \cdots \\ &= \frac{\beta^{(r)}}{r!}x^r\{\nu_r(x,\beta) + \nu_r(x,\beta)\nu_{r+1}(x,\beta) + \cdots\}, \end{aligned}$$

$$\begin{aligned} |B(x,\beta,r)| < \varepsilon\delta\{(1-\delta)+(1-\delta)^2+\cdots\} &= \varepsilon\delta\frac{1-\delta}{1-(1-\delta)} \\ &= \varepsilon(1-\delta) < \varepsilon, \quad \text{for } r > r_2. \qquad (8) \end{aligned}$$

We thus have $\lim_{r\to\infty} B(x,\beta,r) = 0$. Recalling $A(x,\beta,\infty) = A(x,\beta,r) + B(x,\beta,r)$, we have proved that $A(x,\beta,\infty)$ is convergent for $|x| < 1$. This result can be easily extended for $\beta \in \mathcal{R}$. Now,

$$\begin{aligned} (1+x)^\alpha &= \lim_{i\to\infty}(1+x)^{p_i} \\ &= \lim_{i\to\infty}[A(x,p_i,r) + B(x,p_i,r)] \\ &= \lim_{i\to\infty} A(x,p_i,r) + \lim_{i\to\infty} B(x,p_i,r) \\ &= A(x,\alpha,r) + \lim_{i\to\infty} B(x,p_i,r) \text{ for all positive integer } r, \end{aligned}$$

as $A(x,p_i,r)$ consists of finite number of terms.

From result (8),

$$|\lim_{i\to\infty} B(x,p_i,r)| < \varepsilon \text{ for } r > r_2 \text{ since } p_i \in [\alpha-\delta,\alpha+\delta] \text{ for all } i.$$

Since ε can be arbitrary chosen, it follows that $\lim_{r\to\infty}\lim_{i\to\infty} B(x,p_i,r) = 0$.

Recall that for any r, $(1+x)^\alpha = A(x,\alpha,r) + \lim_{i\to\infty} B(x,p_i,r)$. Therefore,

$$\begin{aligned} (1+x)^\alpha &= \lim_{r\to\infty}[A(x,\alpha,r) + \lim_{i\to\infty} B(x,p_i,r)] = A(x,\alpha,\infty) + \lim_{r\to\infty}\lim_{i\to\infty} B(x,p_i,r) \\ &= A(x,\alpha,\infty) = 1 + \frac{\alpha}{1}x + \frac{\alpha^{(2)}}{2!}x^2 + \frac{\alpha^{(3)}}{3!}x^3 + \cdots \end{aligned}$$

for any irrational number α , provided that $|x| < 1$.

The Binomial Theorem is thus generalized to

$$(1+x)^\alpha = 1 + \frac{\alpha}{1}x + \frac{\alpha^{(2)}}{2!}x^2 + \frac{\alpha^{(3)}}{3!}x^3 + \cdots,$$

for all real α , and $|x| < 1$ if $\alpha \notin \mathbb{Z}_0^+$.