Competition Corner

by
Tay Tiong Seng

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Please send your solutions and all other communications about this column to

Dr. Tay Tiong Seng,
Department of Mathematics,
National University of Singapore,
2 Science Drive 2, Singapore 117543

or by e-mail to **mattayts@nus.edu.sg.**

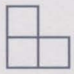
Contributors to this issue are:
A. Robert Pargeter(U.K.), Daniel Chen Chongli(Hwa Chong *Institution),*
Kenneth Tay Jingyi(Anglo-Chinese Junior College), Charmaine Sia Jia Min,
Zhao Yan, Soh Yong Sheng, Wu Jiaweiall from Raffles Junior College
Joel Tay Wei En, Anand B. Rajagopalanand Andre Kueh Ju Luiall from Singapore.

1. (Albanian Mathematical Olympiad, 2000) Prove the inequality

$$\frac{(1+x_1)(1+x_2)\cdots(1+x_n)}{1+x_1x_2\cdots x_n} \le 2^{n-1}, \quad \text{where } x_i \in [1,\infty),\ i = 1,\ldots,n$$

When does equality hold?

2. (Ukrainian Mathematical Olympiad, 2003) Let n be a positive integer. Some $2n^2+3n+2$ cells of a $(2n+1)\times(2n+1)$ square table are marked. Does there always exist one three-cell figure shown below (such figures can be oriented arbitrarily) such that all three cells are marked?



3. (Bulgarian Mathematical Olympiad, 2004) Let I be the incentre of $\triangle ABC$ and let A_1, B_1, C_1 be arbitrary points on the segments AI, BI, CI, respectively. The perpendicular bisectors of AA_1, BB_1, CC_1 intersect at A_2, B_2, C_2. Prove that the circumcentre of $A_2B_2C_2$ coincides with the circumcentre of ABC if and only if I is the orthocentre of $A_1B_1C_1$.

4. (Russian Mathematical Olympiad, 2004) The distance between two 5-digit numbers $\overline{a_1a_2a_3a_4a_5}$ and $\overline{b_1b_2b_3b_4b_5}$ is the maximal integer i for which $a_i \ne b_i$. All the 5-digit numbers are written down one by one in some order. What is the minimal possible sum of distances between adjacent numbers?

5. (Hungarian Mathematical Olympiad, 2002/3) Let n be an integer, $n \ge 2$. We denote by a_n the greatest number with n digits which is neither the sum nor the difference of two perfect squares. (a) Determine a_n as a function of n. (b) Find the smallest value of n for which the sum of squares of the digits of a_n is a perfect square.

6. (Thai Mathematical Olympiad, 2003) Find all primes p such that p^2+2543 has less than 16 distinct positive divisors.

7. (Italian Mathematical Olympiad, 2003/4) Let r and s be two parallel lines and P, Q be points on r and s, respectively. Consider the pair (C_P, C_Q) where C_P is a circle tangent to r at P, C_Q is a circle tangent to s at Q and C_P, C_Q are tangent externally to each other at some point, sat T. Find the locus of T when (C_P, C_Q) varies over all pairs of circles with the given properties.

8. (Estonian Mathematical Olympiad, 2003/4) (a) Does there exist a convex quadrilateral $ABCD$ satisfying the following conditions:

(1) $ABCD$ is not cyclic;

(2) the sides AB, BC, CD and DA have pairwise different lengths;

(3) the circumradii of the triangles ABC, BAD and BCD are equal?

(b) Does there exist such a non-convex quadrilateral?

9. (Indian Mathematical Olympiad, 2004) Let S denote the set of all 6-tuples (a, b, c, d, e, f) of positive integers such that $a^2 + b^2 + c^2 + d^2 + e^2 = f^2$. Consider the set

$$T = \{abcdef : (a, b, c, d, e, f) \in S\}.$$

Find the greatest common divisor of all the members of T.

10. (Ukrainian Mathematical Olympiad, 2004) Let $A_1, A_2, \ldots, A_{2004}$ be the vertices of a convex 2004-gon(i.e., a polygon with 2004 sides). Is it possible to mark each side and each diagonal of the polygon with one of 2003 colours in such a way that the following two conditions hold:

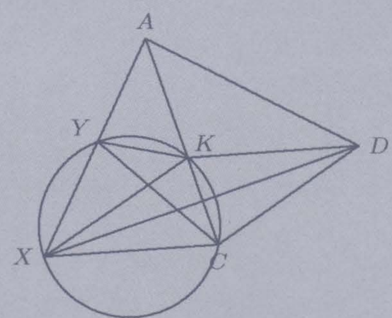
(1) there are 1002 segments of each colour;

(2) if an arbitrary vertex and two arbitrary colours are given, one can start from this vertex and, using segments of these two colours exclusively, visit every other vertex only once?

Solutions to the problems of Volume 31 No.2 2004

1. (Russian Mathematical Olympiad, 2003) Point K is chosen on the diagonal AC of the convex quadrilateral $ABCD$ so that $KD = DC$, $\angle BAC = \frac{1}{2}\angle KDC$, $\angle DAC = \frac{1}{2}\angle KBC$. Prove that either $\angle KDA = \angle BCA$ or $\angle KDA = \angle KBA$.

Solution by Daniel Chen Chongli.



Let the internal angle bisector of $\angle CDK$ meet the line AB at X. Then DX also bisects $\angle KXC$. Also $\angle XAC = \frac{1}{2}\angle KDC = \angle XDC$ and hence $XCDA$ is cyclic. Therefore $\angle DXC = \angle DAC$. Thus $\angle KXC = 2\angle DAC = \angle KBC$. Let the circumcircle of $\triangle XCK$ meet AB at Y. Then $\angle KYC = \angle KXC = \angle KBC$. So X and Y are the only two points on AB that satisfy the property that $\angle KXC = \angle KYC = 2\angle DAC$. Thus B is either X or Y. Since

$$\angle KDA = 90^\circ - \frac{1}{2}\angle KDC - \angle KAD$$

$$\angle KXA = 90^\circ - \angle XAC - \frac{1}{2}\angle KXC$$

$$\angle KXA = \angle YCA$$

we have $\angle KDA = \angle KXA = \angle YCA$. We see that results holds whether $B = X$ or $B = Y$.

2. (Czech and Slovak Mathematical Olympiad, 2003) A sequence $(x_n)_{n=1}^{\infty}$ of integers whose first member $x_1 = 1$ satisfies the condition

$$x_n = \pm x_{n-1} \pm x_{n-2} \cdots \pm x_1$$

for a suitable choice of the signs "+" and "−", for any $n > 1$; for instance, $x_2 = -x_1$, $x_3 = -x_2 + x_1$, $x_4 = x_3 - x_2 - x_1$, ... For a given n, find all possible values of x_n.

Solution by Charmaine Sia Jia Min. Also solved by Daniel Chen Chongli, Joel Tay Wei En, Anand B. Rajagopalan, Kenneth Tay Jingyi, Andre Kueh Ju Lui and Zhao Yan.

It's clear that the set of values for x_1, x_2 and x_3 are, respectively,

$$\{1\}, \{-1, 0, 1\}, \{-2, 0, 2\}.$$

Claim: For $n \geq 3$, the maximum and minimum values of x_n are 2^{n-2} and -2^{n-2}, respectively.

Proof. This is clearly true for $n = 3$. Now we assume that it's true for some $n \geq 3$. Then $\max(x_{n+1}) = \max(x_1) + \max(x_2) + \cdots + \max(x_n) = 1 + 2 + \cdots + 2^{n-2} = 2^{n+1-3}$. The same goes for $\min(x_{n+1}$.

Claim: For $n \geq 3$, by taking $x_i = 2^{i-2}$ for $2 \leq i \leq n-1$, there exists a choice of signs such that x_n can assume any even value in $[-2^{n-2}, 2^{n-2}]$.

Proof. This is clearly true for $n = 3$. We now assume that it's true for some $n = k-1$, $n \geq 3$. Let m be an even integer in $[0, 2^{k-2}]$. Take $x_{k-1} = 2^{k-3}$ which is possible by the induction hypothesis. Note that $m - x_{k-1}$ is an even integer in $[-2^{k-3}, 2^{k-3}]$. By the induction hypothesis again, we can choose $x_1, \ldots, x_{k-2}$ so that $\pm x_1 \pm \cdots \pm x_{k-2} = m - x_{k-1}$. Thus $x_{k-1} \pm x_{k-2} \pm \cdots \pm x_1 = m$. Thus the value m can be attained. By reversing the signs in the above expression, we can also obtained $-m$. Thus the claim is complete.

3. (Hong Kong Mathematical Olympiad, 2003) Let $n \geq 3$ be an integer. In a conference there are n mathematicians. Every pair of mathematicians communicate in one of n official languages of the conference. For any three different official languages, there exist three mathematicians who communicate with each other in these three languages. Determine all n for which this is possible. Justify your claim.

Solution by Charmaine Sia Jia Min. Also solved by Daniel Chen Chongli.

Since there $\binom{n}{3}$ triples of languages and the same number of triples of mathematicians, there is a bijection between these two types of triples. It's clear that the number of mathematicians speaking each language must be the same. Hence $n \mid \binom{n}{2}$, i.e., $2 \mid n-1$ and n is odd.

Now we consider the following construction: colour each of the edges of a regular polygon with n sides, n odd, is a different colour. Now colour all the diagonals of the polygon which are parallel to a particular edge in the same colour as that edge. Since the triple of gradients (a, b, c) of the sides of any triangle in the polygon is unique and there are $\binom{n}{3}$ ways of choosing the gradients for a triangle and $\binom{n}{3}$ ways of choosing a triangle, there is a bijection between every combination of 3 colours and every triangle.

4. (Romanian Mathematical Olympiad, 2003) An integer n, $n \geq 2$ is called *friendly* if there exists a family $A_1, A_2, \ldots, A_n$ of subsets of the set $\{1, 2, \ldots, n\}$ such that:

(1) $i \notin A_i$ for every $i \in \{1, 2, \ldots, n\}$;

(2) $i \in A_j$ if and only if $j \notin A_i$, for every distinct $i, j \in \{1, 2, \ldots, n\}$;

(3) $A_i \cap A_j$ is nonempty, for every $i, j \in \{1, 2, \ldots, n\}$.

Prove that n is friendly if and only if $n \geq 7$.

Solution by Soh Yong Sheng. We consider the $n \times n$ *incidence matrix* of the sets. This is the matrix where the rows are indexed by the sets $A_1, A_2, \ldots, A_n$ and the columns are indexed by $1, 2, \ldots, n$. The (A_i, j) entry of the matrix is 1 if $j \in A_i$ and is 0 otherwise. Conditions 1 and imply that the matrix is "anti-symmetric" in the sense that the main diagonal consists of 0's and the for all $i \neq j$, the (i, j) and (j, i) entries are different. Condition 3 implies that every pair of rows must have a 1 in the corresponding position. The following incidence matrix shows that $n = 7$ is friendly:

$$\begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \\ A_7 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

For $n > 7$ the following $n \times n$ matrix is friendly:

$$\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$$

where X is the above 7×7 friendly matrix, Y is the matrix with the second row consisting of 1's while all the other entries are 0's, Z is the matrix with the second column consisting of 0's while all the other entries are 1's and Z is an arbitrary antisymmetric matrix.

Finally we show that n is not friendly for $n \leq 6$. Since the total number of 1's in the matrix is $\binom{n}{2}$, and that $3n > \binom{n}{2}$ for $n \leq 6$, there is a row with at most two 1's. Without loss of generality, let $A_1 = \{2, 3\}$. Then we see that $3 \in A_2$ and $2 \in A_3$ by condition 3. But this violates condition 2. Thus n is not friendly.

5. (Russian Mathematical Olympiad, 2003) Let $f(x)$ and $g(x)$ be polynomials with nonnegative integer coefficients and that m is the largest coefficient of f. It is known that for some natural numbers $a < b$, the equalities $f(a) = g(a)$ and $f(b) = g(b)$ are true. Prove that if $b > m$, then f and g are identical.

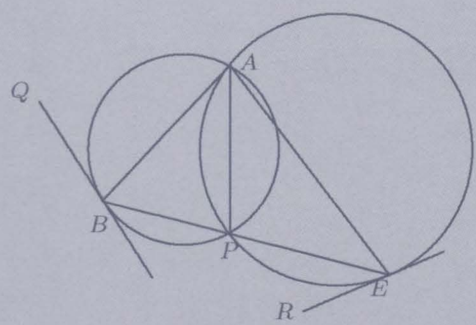
Solution by Wu Jiawei. Suppose $f(x) \not\equiv g(x)$. Let

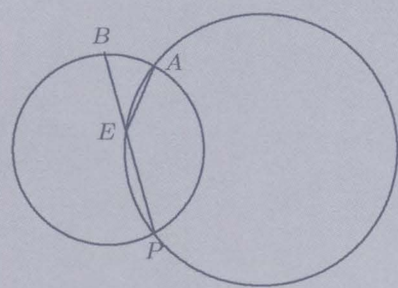
$$f(x) = \alpha_p x^p + \cdots + \alpha_1 x + \alpha_0$$
$$g(x) = \beta_q x^q + \cdots + \beta_1 x + \beta_0$$

Since $m < b$, when written in base b, we have $f(b) = (\overline{\alpha_p \ldots \alpha_1 \alpha_0})_b$. There exists j such that $\beta_j \geq b$, for if not, then by the uniqueness of base b representation, we have $p = q$ and $\alpha_j = \beta_j$ for all j, i.e. $f(x) \equiv g(x)$. Let i be the smallest index such that $\beta_i \geq b$. Let $\beta_i = kb + r$, k, r are integers with $0 \leq r < b$ and $k \geq 1$. Let $g_1(x) = \beta'_q x^q + \cdots + \beta'_1 x + \beta'_0$, where $\beta'_i = r$, $\beta'_{i+1} = \beta_{i+1} + k$ and $\beta'_j = \beta_j$ for all other j's. Then $g(b) = g_1(b)$ and $g_1(a) - g(a) = ka_i(b - a) > 0$. Repeat this process say k times, we obtain a sequence of polynomials $g_1(x), \ldots, g_k(x)$, where the coefficients of g_k are all nonnegative and less than b, $g(b) = g_1(b) = \cdots = g_k(b) = f(b)$ and $f(a) = g(a) > g_1(a) > \cdots > g_k(a)$. But from $g_k(b) = f(b)$, we get $g_k(x) \equiv f(x)$. Thus we get a contradiction and so $f(x) \equiv g(x)$.

6. (Austrian Mathematical Olympiad, 2002) Let $ABCD$ and $AEFG$ be similar cyclic quadrilaterals, whose vertices are labeled counterclockwise. Let P be the second common point of the circumcircles of the quadrilaterals beside A, show that P must lie on the line connecting B and E.

Similar solutions by Joel Tay Wei En, Kenneth Tay Jingyi, Anand B. Rajagopalan, Andre Kueh Ju Lui, Daniel Chen Chongli, Zhao Yan, Soh Yong Sheng and A. Robert Pargeter who also points out the result holds true for any pair of similar cyclic polygon.





There are two cases. The first being that E is outside the circumcircle of $ABCD$. The similarity of the polygons implies that $\angle ABQ = \angle AER$. The alternate segment theorem then implies that $\angle APB + \angle APE = 180°$. So B, E, P are collinear. The second is that E is inside the circumcircle of $ABCD$. In this case we have $\angle APB = \angle APE$, whence B, E, P are collinear.

7. (Belarusian Mathematical Olympiad, 2003) Given a convex pentagon $ABCDE$ with $AB = BC$, $CD = DE$, $\angle ABC = 150°$, $\angle CDE = 30°$, $BD = 2$, find the area of $ABCDE$.

Similar solution by Anand B. Rajagopalan and Zhao Yan. Also solved by Daniel Chen Chongliand Charmaine Sia Jia Min.

Let $AB = BC = a$, $CD = DE = b$ and $\angle ACE = x$. By the cosine rule on BCD,

$a^2+b^2-2ab\cos(90+x)=a^2+b^2+ab\sin x=4$. Thus

$$\begin{aligned}[ABCDE] &= [ABC]+[ACE]+[ECD] \\ &= (a^2\sin 15+b^2\sin 30+(2a\cos 75)(2b\cos 15)\sin x)/2 \\ &= \frac{1}{2}\sin 30(a^2+b^2+2ab\sin x)=1.\end{aligned}$$

8. (Korean Mathematical Olympiad, 2003) Show that there exist no integers x, y, z satisfying

$$2x^4+2x^2y^2+y^4=z^2, \quad x\neq 0. \tag{1}$$

Solutions by Wu Jiawei. Suppose the equation has a solution. Rewrite it as

$$(x^2+y^2)^2+(x^2)^2=z^2.$$

Let x be the least positive integer that satisfies the equation. Suppose $\gcd(x,y)=a$. Then by dividing both sides by a^4 we see that x/a also satisfies the equation. Thus $a=1$. By Pythagorean triples, we get two cases: Case 1:

$$x^2 = 2ab \tag{2}$$
$$x^2+y^2 = a^2-b^2 \tag{3}$$
$$z^2 = a^2+b^2 \tag{4}$$

or case 2:

$$x^2 = a^2-b^2 \tag{5}$$
$$x^2+y^2 = 2ab \tag{6}$$
$$z^2 = a^2+b^2 \tag{7}$$

In case 2, we have from (6), that x and y are of the same parity. Thus they must be odd. From (5), a and b are of opposite parity. Thus taking mod 4 in (6) we get $2\equiv 0$, a contradiction. Thus case 2 is impossible.

In case 1, suppose $\gcd(a,b)=k$. Then $k\mid x$ from (2) and $k\mid y$ from (3). Thus $k=1$. Since $\gcd(y,a)\mid x$ and $\gcd(y,b)\mid x$, we see that $\gcd(y,a)=\gcd(y,b)=1$. From (2), x is even. Since $\gcd(x,y)=1$, y is odd. From (3), we have $a^2-b^2\equiv 1 \pmod 4$. Hence b is even and a is odd. Since $\gcd(a,b)=1$, we have, from (2), $a=c^2$, $b=2d^2$ for some positive integers c,d with $\gcd(c,d)=1$, and c odd. Then

$$x^2 = 4c^2d^2 \tag{8}$$
$$y^2 = c^4-4c^2d^2-4d^4 \tag{9}$$

Let $k=c^2-2d^2$. Then k is odd. We have

$$k^2-y^2=8d^4, \quad \text{i.e.} \quad \left(\frac{k}{2}+\frac{y}{2}\right)\left(\frac{k}{2}-\frac{y}{2}\right)=2d^4.$$

Let $A = \frac{k}{2} + \frac{y}{2}$ and $B = \frac{k}{2} - \frac{y}{2}$. Then A and B are both integers as k and y are both odd. Then $A - B = y$ and $AB = 2d^4$. Since $\gcd(y, 2d^4) = 1$ (from (9)), if p is a prime that divides two of d, A and B, then it can't divide the third. Thus if A is even, then there exist and coprime positive integers p, q such that $A = 2p^4$ and $B = q^4$. Then

$$2p^4 + 2p^2q^2 + q^4 = k + 2d^2 = c^2.$$

Since $p \leq d < x$, it contradicts the minimality of x. So case 1 is also impossible. Thus the equation has no solution.

9. (Iranian Mathematical Olympiad, 2003) Find the smallest positive integer n such that:

"For any finite set of points in the plane, if for every n points of this set, there exist two lines covering all n points, then there exist two lines covering all the set."

Similar solutions by Anand B. Rajagopalan and Andre Kueh Ju Lui. Also solved by Daniel Chen Chongli, Joel Tay Wei En and Zhao Yan.

The following configuration shows that $n \geq 6$ since every set of five points can be covered by 2 lines but the entire set cannot be covered by two lines..

We shall now show that $n = 6$. Take any set of 6 points. Since they are covered by 2 lines, one of the lines , say ℓ, must contain 3 points, say A, B, C. If there are at most 2 points not covered by ℓ, then two lines can cover all the points. So suppose there are three points X, Y, Z not on ℓ. Since $\{A, B, C, X, Y, Z\}$ can be covered by two lines, one of the lines must be ℓ for otherwise we would need three lines to cover A, B, C. So X, Y, Z must be covered by a single line m. This argument can be repeated to show that all the points not covered by ℓ are covered by m and so the proof is complete.

10. (Finland Mathematical Olympiad, 2003)

Players Aino and Eino take turns in choosing numbers from the set $\{0, 1, \ldots, n\}$, where $n \in \mathbb{N}$ is a predetermined number. The game ends when the numbers chosen by either player contain a set of four numbers that can be arranged to an arithmetic progression. The player whose numbers contain such progression wins the game. Show that there exists an n for which the first player has a winning strategy. Find the least possible n for which this is possible.

Solution by Daniel Chen Chongli. Also solved by Andre Kueh Ju Lui.

For $n = 14$, there is a winning strategy for the first player, say Aino. Aino takes 7. By symmetry, we need only consider what happens if Eino takes $0, 1, \ldots, 6$. If Eino takes 0, 1, 2, 3, or 4, Aino takes 9, threatening to take 8 and win at either 6 or 10. This Eino must take 6, 8 or 10. Now Aino takes 11 and will win at 5 or 13.

If Eino takes 5, Aino takes 10, threatening to take 4 and win at 1 or 13. If Eino takes 1 or 13, Aino takes 8, Eino must take 9 and Aino takes 12 and wins with either 6 or 14. If Eino tales 4, Aino takes 8, Eino must take 9, Aino takes 12 and wins with either 6 or 14.

The case Eino takes 6 is similar and is omitted.