Mathematical Medley Problems Corner

A Prized Problems

Problem 1

(Book voucher up to $150)

(proposed by Wong Fook Sung Albert, Temasek Polytechnic)

Find the exact value of the sum

$$\sum_{n=1}^{\infty} \frac{1}{(2n+1)(4n+1)}$$

Problem 2

(Book voucher up to $150)

Let $n \geq 4$ and A be a set of n integers in the open interval $(0, 2n)$.

Can we always find elements in A whose sum is divisible by $2n$?

B Instruction

1. Prizes in the form of book vouchers will be awarded to one or more received best solutions submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

3. Solutions should be sent to : The Editor, Mathematics Medley
c/o Department of Mathematics
National University of Singapore
2 Science Drive 2
Singapore117543

   and should arrive before **1 March 2005**.

   Alternatively, softcopies of the solutions can also be sent to the email address: mattanv@nus.edu.sg.

4. The Editor's decision will be final and no correspondence will be entertained.

Solutions to the problems of volume 31, No1, 2004

Problem 1

A cubic block is partitioned into n^3 unit cubic blocks. Two unit blocks are adjacent if they share a common face. Two players A and B play a game as follow:

A starts from any unit block and moves to an adjacent unit block. Then B moves from this new block to another adjacent block. The two players move alternately without revisiting any unit block. The first player who cannot move on lose the game.

Suppose both players are playing with their best strategies. Who will win the game?

Solution 1 to Problem 1

By Shawn Eastwood – Canadian International School (Singapore)

Let n be even.

Player A imagines the n by n by n cube partitioned into blocks that are 1 by 1 by 2, or collections of pairs of cells that are adjacent. When it is player A's turn he moves into the other half of the pair. Player B has no choice but to leave the pair into another pair, which could not be revisited. Hence any pair visited will not have been entered before and any pair left behind could not be visited again. Hence player A will always have somewhere to move on his turn: the other half of the pair. Hence player A always wins with his best strategy.

Now let n be odd.

We alternately colour each cell black or white starting with one corner coloured black so that every black cell has white cells for neighbours and every white cell has black cells for neighbours (The colouring of a 3D chess board). Players start their move on always the same colour for the duration of the game. There is one more black cell than white cell, and because each pair covers both a black cell and a white cell, there will always be one black cell left over. Now player A wants to effectively remove this cell from the game. Since this black cell is surrounded by white cells, the player who moves from the white cells is the only player that could move into it. Hence player A chooses to move from a white cell to start the game, then the one black cell remaining will never be entered by player A because it is not contained in any pair. Hence player A always wins with his best strategy.

For any value of n, player A always wins with his best strategy

Solution 2 to Problem 1

By Charmaine Sia Jia Min - Raffles Junior College

A will win the game.

Colour the unit blocks black and white in chessboard style such that adjacent unit blocks are of different colours.

Case 1: n is even.

We may group the n^3 unit blocks into $n^3/2$ non-intersecting "dominos" (formed by 2 adjacent blocks). Let A start from a white block and move to the black block of the same "domino". Then B has to move (if possible) to the white block of an untouched "domino". A can then move to the black block of the same "domino".

The process repeats. Every time A has completed his turn in this manner, B has to move to the white block of an untouched "domino". Hence, A can always move after B has moved, and hence B will lose.

Case 2: n is odd.

Without loss of generality, we may assume that the unit blocks at the corners are black. Now, "hide" one of the black blocks at a corner. We may then group the remaining $n^3 - 1$ unit blocks into $(n^3 - 1)/2$ non-intersecting "dominos". A starts from a white block and follows the process mentioned in Case 1. Since B always moves into a white block, B cannot move into the "hidden" block and A will not move into the "hidden" block since it is not part of a domino. By the same argument as in Case 1, A can always move after B has moved and hence B will lose.

Editor's note:
The book prize of $150 is shared between Shawn and Charmaine.

Problem 2

Let A be a set of k natural numbers.
Denote by $A+A$ *the set* $\{n : n = a + b \text{ where } a, b \in A\}$.
Show that $A + A$ has $2k - 1$ (distinct) elements if and only if A contains an arithmetic progression of length k.

Solution 1 to Problem 2

By Hang Hao Chuien -The Chinese High School

Let $A = \{a_1, a_2, a_3, a_4, \ldots, a_k\}$ where $a_i \in$ the set of natural numbers, $i = 1, 2, \ldots, k$ and $a_1 < a_2 < a_3 < a_4 < \ldots < a_k$.

If $A + A$ has $2k - 1$ distinct elements, then the elements of $A + A$ are given by

$$a_i + a_j, \quad 1 \le i \le j \le k.$$

These are represented as follows:

$$\begin{array}{l} 2a_1, a_1 + a_2, a_1 + a_3, a_1 + a_4, \ldots, a_1 + a_k, \\ \quad 2a_2, a_2 + a_3, a_2 + a_4, \ldots, a_1 + a_k, \\ \quad\quad \ldots\ldots\ldots\ldots\ldots\ldots \\ \quad\quad\quad \ldots\ldots\ldots\ldots \\ \quad\quad\quad\quad 2a_{k-1}, a_{k-1} + a_k, \\ \quad\quad\quad\quad\quad 2a_k \end{array}$$

Since $a_1 < a_2 < a_3 < a_4 < \ldots < a_k$, we have

$$2a_1 < a_1 + a_2 < 2a_2 < a_2 + a_3 < 2a_3 < \ldots < 2a_{k-1} < a_{k-1} + a_k < 2a_k.$$

These, therefore, make up the $2k - 1$ distinct elements of $A + A$ which can be represented either as $2a_i$, $i = 1, 2, \ldots, k$, or $a_j + a_{j+1}$, $j = 1, 2, \ldots, k-1$.
We also know that for $i = 1, 2, \ldots, k - 2$, $a_i + a_{i+1} < a_i + a_{i+2} < a_{i+1} + a_{i+2}$.
But $a_i + a_{i+1} < 2a_{i+1} < a_{i+1} + a_{i+2}$.
As there are only $2k - 1$ distinct elements in $A + A$,
we have $2a_{i+1} = a_i + a_{i+2}$ for $i = 1, 2, \ldots, k - 2$.
Hence, the k elements in A form an arithmetic progression of length k.

Conversely, if the elements of A form an arithmetic progression of length k, then $a_i = a_1 + (i-1)d$, where d is a positive integer, $i = 1, 2, \dots, k$.
Thus, each element in $A + A$ is of the form $2a_1 + (i + j - 2)d$, where $1 \le i \le j \le k$.
But $2 \le i + j \le 2k$. Hence, $A + A$ has $2k - 1$ distinct elements.

Solution 2 to Problem 2

By Kenneth Tay Jingyi - Anglo-Chinese Junior College

Let $A = \{a_1, a_2, \dots, a_k\}$ with $a_1 < a_2 < \dots a_k$.
Let $b_{ij} = a_i + a_j$ with $i \le j$. Then $A + A = \{b_{ij} \mid i, j = 1,2,\dots, k, i \le j\}$.
First assume that $A + A$ has exactly $2k$-1 elements. We want to show that the a_i's form an arithmetic progression. Now

$$i_1 \le j_1, i_2 \le j_2 \Rightarrow a_{i_1} + a_{i_2} \le a_{j_1} + a_{j_2} \Rightarrow b_{i_1 i_2} \le b_{j_1 j_2}$$

Equality holding in the last inequality only when $i_1 = j_1$, $i_2 = j_2$
Note that $b_{11} < b_{12} < b_{22} < b_{23} < b_{33} < \dots < b_{kk}$. This gives us $2k$-1 distinct elements in $A + A$. Thus $A + A = \{b_{11}, b_{12}, b_{22}, b_{23}, b_{33}, \dots, b_{kk}\}$
However, $b_{11} < b_{12} < b_{13} < b_{14} < \dots < b_{1k} < b_{2k} < b_{3k} < \dots < b_{kk}$. This gives us $2k$-1 distinct elements in $A + A$ as well.

$\Rightarrow b_{11} = b_{11}, b_{12} = b_{12}, b_{13} = b_{22}, b_{14} = b_{23}, \dots, b_{kk} = b_{kk}$

Alternatively, $b_{1i} = b_{\left\lfloor \frac{i+1}{2} \right\rfloor \left\lceil \frac{i+1}{2} \right\rceil}$ for i = 1, 2,…, k.

Let $a_i = X$, $a_2 = X + Y$. Now we show by induction on i that $a_i = X + (i-1)Y$.
The induction statement is clearly true for $i = 1$ and 2.
Assume that $a_i = X + (i-1)Y$ for $i = 1, 2, \dots, m$ for some $m \ge 1$.
We know that $b_{1(m+1)} = b_{\left\lfloor \frac{m+2}{2} \right\rfloor \left\lceil \frac{m+2}{2} \right\rceil}$.

If m is even, let $m = 2h$. Then $b_{1(2h+1)} = b_{(h+1)(h+1)}$.

$a_1 + a_{2h+1} = a_{h+1} + a_{h+1} \Rightarrow X + a_{2h+1} = X + hY + X + hY \Rightarrow a_{2h+1} = X + 2hY$
$\Rightarrow a_{m+1} = X + mY$

If m is odd, let $m = 2h+1$. Then $b_{1(2h+2)} = b_{(h+1)(h+2)}$

$a_1 + a_{2h+2} = a_{h+1} + a_{h+2} \Rightarrow X + a_{2h+2} = X + hY + X + (h+1)Y \Rightarrow a_{2h+2} = X + (2h+1)Y$
$\Rightarrow a_{m+1} = X + mY$

By induction we have proved the induction hypothesis, thus the a_i's form an arithmetic progression.

Conversely, if the a_i's form an arithmetic progression, let $A = \{X + iY \mid i = 0,1,2, \dots, k-1\}$. Then $A + A = \{2X + iY \mid i = 0,1,2, \dots, 2i - 2\}$. So $A + A$ has exactly $2k$-1 elements.

Editor's note:
Also solved correctly by Charmaine Sia, Shawn Eastwood, Ong Xing Cong from Raffles Junior College and Cong Lin from Hwa Chong Junior College. The book prize of $150 is shared between Hao Chuien and Kenneth