PASCAL'S TRIANGLE IN 2-D PALINDROMIC SEQUENCES

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1. Motivation

Palindromes are words, phrases or numbers that spell the same backwards as forwards. For example, the phrase:

"Madam, I'm Adam"

is a palindrome. Existence of symmetry is, perhaps, the reason why palindromes are objects of beauty. Palindromes have been investigated over the years in terms of (a) obtaining palindromic numbers via arithmetical operation [1-5], (b) generating palindromic primes [6,7] and (c) the study of numerical palindromic properties [8-11]. In recent years the subject of palindromy has been investigated in the areas of computational mathematics for analyzing languages [12], biology for understanding cell replication [13] and chemical analysis of DNA [14]. In this note we shall see a special class of palindromes derived from palindromes. These palindromes can be expressed as a function, say, $f(m)$, and upon substitution of positive integers $m = 1, 2, 3, \ldots$ into $f(m)$ gives rise to a sequence of palindromes. Here, we consider a class of palindromes expressible as a function of two variables, say, $f(m, n)$, whereupon substitution of $m, n = 1, 2, 3, \ldots$ gives a group of palindromes which, being 2-dimensional sequences, can be best appreciated in tabular forms. To make things more interesting, we narrow down our analysis to a special class of 2-dimensional palindromic sequences which exhibit Pascal's Triangle. The Pascal's Triangle, as shown in Figure 1, is useful for (a) extracting coefficients of binomial expansion, (b) obtaining the number of ways of choosing r objects from a number of n objects, and (c) obtaining Fibonacci's sequence, among various applications.

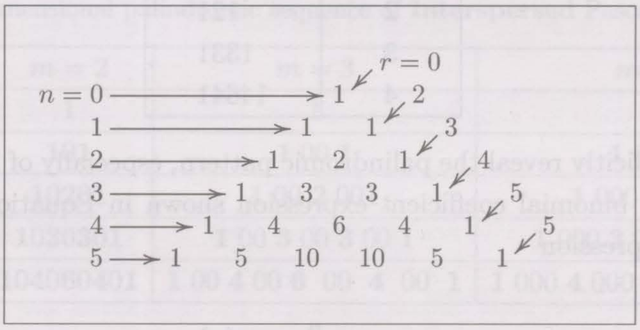

Figure 1. Pascal's Triangle up to $n = 5$

Pascal's Triangle in 2-D Palindromic Sequences

The relevance of Pascal's Triangle is undoubted and its literature too numerous to be listed. Interested readers are referred to the following literature: [15-21]. Here, we intend to enjoy the Pascal Palindromy as a type of number pattern for aesthetical purposes [22]. Since the sequence

$$\binom{n}{r} = \frac{n!}{r!(n-r)!} \tag{1}$$

has a mirror image at $r = (n/2)$ for $n =$ even, and between two elements $r = (n-1)/2$ and $r = (n+1)/2$ for $n =$ odd, then the sequence described by Equation (1) for integer $r \in [0, n]$ is a palindromic sequence. As such, multiplying $\binom{n}{r}$ with 10^r or 10^{n-r} for nonnegative integer sequence of r up to $n = 4$ gives palindromic patterns because each consecutive element is placed at an order higher (for $\times 10^r$) or an order lower (for $\times 10^{n-r}$) than the previous element. Therefore this special class of palindromes, herein named "Pascal's Palindrome", is obtained by combining every element for each row in the Pascal's Triangle to form one integer, i.e. 1, 11, 121, 1331, 14641. A generalized expression is obtained by writing the Pascal's Palindromic function as 11^n for $n = 0, 1, 2, 3, 4$, as shown in Table 1.

Table 1. Pascal Palindromes

n	11^n
0	1
1	11
2	121
3	1331
4	14641

In order to explicitly reveal the palindromic pattern, especially of Pascal's elements, we employ the binomial coefficient expression shown in Equation (1) to give the palindromic expression

$$11^n = \sum_{r=0}^{n} 10^r \binom{n}{r} \tag{2a}$$

which reads backwards from right to left, or

$$11^n = \sum_{r=0}^{n} 10^{n-r} \binom{n}{r} \tag{2b}$$

which reads forward from left to right, valid for $n = 0, 1, 2, 3, 4$.

Pascal's Triangle in 2-D Palindromic Sequences

2. Interspersed Pascal Palindromy

Introducing a palindromic function of the form

$$f(m) = 11, 101, 1001, \ldots = 10^m + 1,$$

then taking positive integer powers for $f(m)$ gives

$$g(m,n) = [f(m)]^n = (10^m + 1)^n, \tag{3}$$

Applying binomial theorem, Equation (3) can be written as

$$g(m,n) = \sum_{r=0}^{n} 10^{mr} \binom{n}{r} \tag{4a}$$

Equation (4a) is a palindrome as implied by Equation (1). By virtue of Equations (2a) and (2b), the backward representation in Equation (4a) can be read forward when written as

$$g(m,n) = \sum_{r=0}^{n} 10^{m(n-r)} \binom{n}{r}. \tag{4b}$$

The palindromic patterns in Equations (4a) and (4b) are valid for all positive integers m but limited for integer $n \in [0,4]$. This may well be seen from Figure 1 where for rows $n \geq 5$, there exists double digits in some elements, thereby upsetting the palindromic pattern. The 2-dimensional palindromic sequence described by $g(m,n)$ is displayed in Table 2 up to $m = n = 4$, whereby elements from Pascal's Triangle are bolded for clarity.

Table 2. Two-dimensional palindromic sequence of **Interspersed** Pascal Palindrome

	$m=1$	$m=2$	$m=3$	$m=4$
$n=0$	**1**	**1**	**1**	**1**
$n=1$	**11**	101	1 00 1	1 000 1
$n=2$	**121**	10201	1 00 2 001	1 000 2 000 1
$n=3$	**1331**	1030301	1 00 3 00 3 00 1	1 000 3 000 3 000 1
$n=4$	**14641**	104060401	1 00 4 00 6 00 4 00 1	1 000 4 000 6 000 4 000 1

We note that for each n, there exist $(m-1)$ zeros present between the Pascal Triangle's elements since the terms in Equation (4) increment in the order of m. For each m, progression of n reflects the progression of Pascal's Triangle by the horizontal rows as implied by the term $\binom{n}{r}$ in Equation (4). Since elements of Pascal's Triangle are distributed with uniformly spaced interval, the 2-dimensional palindromic sequence described by $g(m,n)$ is hereby termed the Interspersed Pascal Palindromy.

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3. Twin Pascal Palindromy

Consider now the function

$$h(m,n) = 11^n f(m) = (10^m + 1)\sum_{r=0}^{n} 10^n \binom{n}{r} \tag{5}$$

which is a product of two palindromic terms. The RHS of Equation (5) can be split into two terms,

$$h(m,n) = \sum_{r=0}^{n} 10^r \binom{n}{r} + 10^m \sum_{r=0}^{n} 10^r \binom{n}{r}, \tag{6a}$$

whereby each term on the RHS displays Pascal's Palindromy, reading backwards. For forward reading, Equation (6a) can be written in its equivalence form

$$h(m,n) = 10^m \sum_{r=0}^{n} 10^{n-r} \binom{n}{r} + \sum_{r=0}^{n} 10^{n-r} \binom{n}{r}. \tag{6b}$$

Results of $h(m,n)$ is shown in Table 3 wherein each of the Twin Pascal Palindromes is underlined.

Table 3. Two-dimensional palindromic sequence of **Twin** Pascal Palindromes

	$m=1$	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$
$n=0$	1 1	101	1001	10001	100001	1000001
$n=1$	121	11 11	11011	110011	1100011	11000011
$n=2$	1331	12221	121 121	1210121	12100121	121000121
$n=3$	14641	134431	1332331	1331 1331	133101331	1331001331
$n=4$	–	1478741	14655641	146424641	14641 14641	14641014641

For sub-palindromes to exist in Equation (6b), the last term of $10^m \sum_{r=0}^{n} 10^{n-r}\binom{n}{r}$, being lowest in value, must be at least one order higher than the first term of $\sum_{r=0}^{n} 10^{n-r}\binom{n}{r}$, being the highest in value. That is, $m \geq (n+1)$ as evident from Table 3. We further note that

(a) palindromic pattern is observed for all m, but

(b) this palindromic pattern is limited to $n=3$ (for $m=1$) and $n=4$ (for $m>1$), and that

(c) twin palindromes (shown underlined in Table 3) is observed for $m \geq (n+1)$.

Where $m \leq n$, palindromic pattern persist, albeit without the twin palindromes. To better appreciate the beauty of twin palindromy, let us look again at the phrase, "Madam, I'm Adam." Removing all punctuation marks and writing in uppercase, we have

M A D A M I M A D A M.

As shown by the underlinings, this palindromic phrase consists of two sub-palindromes, each of which is spelt

M A D A M.

Since these sub-palindromes exist as an identical pair, we call this palindromic phenomenon as Twin Palindromy.

4. Conclusion and Further Motivation

A special class of palindromes which are derived from other palindromes, which display 2-dimensional palindromic sequences, and which exhibits elements from Pascal's Triangle, has been demonstrated in this note. Two sets of such palindromes have been looked into:

(a) where Pascal Triangle's elements are interspersed amongst zeroes, and
(b) where Pascal Triangle's elements occur as twins, distinctly separated by zeroes.

The twin palindrome illustrated herein can, in fact, be classified under the category of a two-ordered palindrome whereby the entire palindrome itself is of the first order and the sub-palindrome is the second order. As an example, the palindromic phrase depicted in Figure 2 reveals two sub-palindromes which, in turn, consist of four sub-sub-palindromes.

Madam, I'm Adam: Mad am I, ma? DAM!

||

MADAM I MADAM MADAM I MADAM

3rd order 3rd order 3rd order 3rd order

2nd order 2nd order

1st order

Figure 2. Alphabetical example of palindrome in 3 orders

Hence we may call this a three-ordered palindrome. As such, obtaining numerical palindromes of three or more orders, consisting significant numbers (such as Pascal Triangle's elements, plateau primes, palindromic primes, etc) as sub-palindromes, is recommended for future analysis and appreciation.

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