CONCURRENT CEVIANS TO THE HALF-WAY POINTS OF A TRIANGLE

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Let us consider the triangle shown in Figure 1. Its vertices are denoted by A, B, and C, and the lengths of sides AB, BC and CA are denoted in the standard manner as c, a and b, respectively.

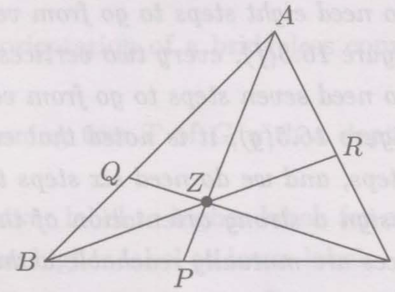

Figure 1. △ ABC with Cevians to the half-way points

Point Q lies on AB and has the property that $CA + AQ = CB + BQ = (a+b+c)/2$. Point P lies on BC and has the property that $AB + BP = CA + CP = (a+b+c)/2$. Point R lies on CA and has the property that $AB+AR = BC+CR = (a+b+c)/2$. We call P, Q and R the "half-way points" in $\triangle ABC$ opposite to A, C and B, respectively.

Concurrent Cevians. We have drawn cevians AP, BR, and CQ as though they are concurrent. Let us prove that in fact they are.

It should be clear that $AR = (a+b+c)/2 - c = (a+b-c)/2$ and $CR = (a+b+c)/2 - a = (-a+b+c)/2$. Similarly $CP = (a+b+c)/2 - b = (a-b+c)/2$ and $BP = (a+b+c)/2 - c = (a+b-c)/2$. Also $BQ = (a+b+c)/2 - a = (-a+b+c)/2$ and $AQ = (a+b+c)/2 - b = (a-b+c)/2$.

The obvious way to proceed is by means of Ceva's Theorem. However, we shall use convex (or barycentric) coordinates to complete our proof since their use makes possible several interesting computations. We note in passing that Ceva's Theorem itself can be proved with convex coordinates.

Each point Z of the closed triangular region ABC has uniquely associated with it in a one-to-one manner an ordered triple of real numbers (α, β, γ) where $\alpha \geq 0$, $\beta \geq 0$, $\gamma \geq 0$, and $\alpha + \beta + \gamma = 1$. These three numbers, α, β, and γ, are the convex coordinates of Z with respect to the vertices A, B, and C in that order. The convex coordinates of Z may be interpreted as the distribution of 1 unit of weight, α at A, β at B, and γ at C, which defines Z to be the balance point of the otherwise weightless triangular region.

As examples, the convex coordinates of A, B, C, and the centroid of the triangular region are $(1,0,0)$, $(0,1,0)$, $(0,0,1)$, and $(1/3,1/3,1/3)$, respectively. The convex coordinates of the midpoints of AB, BC, and CA are $(1/2, 1/2.0)$, $(0, 1/2, 1/2)$, and $(1/2, 0, 1/2)$, respectively.

Let us place weights α at A, β at B, and γ at C so that cevian CQ will fall on a balance line of the region. Since γ at C is supported by the cevian, we have balance if and only if $(AQ)\beta = (BQ)\alpha$ or

$$\beta(a-b+c)/2 = \alpha(-a+b+c)/2. \quad (1)$$

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Similarly, cevian BR will define a balance line if and only if $(AR)\gamma = (CR)\alpha$ or

$$\gamma(a+b-c)/2 = \alpha(-a+b+c)/2. \qquad (2)$$

Cevian AP will define a balance line if and only if $(BP)\gamma = (CP)\beta$ or

$$\gamma(a+b-c)/2 = \beta(a-b+c)/2. \qquad (3)$$

All three cevians will define balance lines if and only if there exists a triple of numbers α, β and γ that satisfies equations 1, 2, and 3 simultaneously. If a solution exists, the balance point for the weight distribution must be located on each cevian. Hence, the three cevians must be concurrent at the balance point. If the numbers α, β and γ also satisfy the equation $\alpha+\beta+\gamma = 1$ with $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \geq 0$, the three numbers are the convex coordinates of the balance point.

Such a solution does exist. It is given by

$$(\alpha,\beta,\gamma) = \left(\frac{(b-c)^2-a^2}{a^2+(b-c)^2-2a(b+c)}, \frac{(a-c)^2-b^2}{a^2+(b-c)^2-2a(b+c)}, \frac{(a-b)^2-c^2}{a^2+(b-c)^2-2a(b+c)}\right). \qquad (4)$$

Thus the cevians to the half-way points of the triangle are concurrent and the convex coordinates of the point of concurrence are given by equation 4.

Next, we give two calculations which make use of the convex coordinates which we have found.

Example 1. Isosceles triangle ABC is located in the Cartesian plane. Vertices A, B and C have Cartesian coordinates $(-5,0)$, $(0,12)$ and $(5,0)$, respectively. Let us find the Cartesian coordinates of point Z at which the cevians to the half-way points of the triangle are concurrent.

Solution. Applying the distance formula, we find that $a = 13$, $b = 10$ and $c = 13$. From equation 4, we find that the convex coordinates of Z are $(\alpha,\beta,\gamma) = (8/21, 5/21, 8/21)$. The desired Cartesian coordinates (x,y) of Z in $\triangle ABC$ are given by convex combinations of the Cartesian coordinates of the vertices of the triangle. Thus,

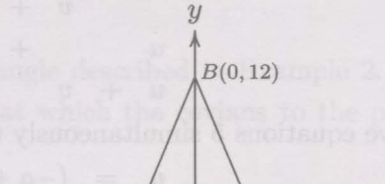

Figure 2. The Isosceles Triangle

$$x = (-5)\alpha + (0)\beta + (5)\gamma = (-5)\frac{8}{21} + (0)\frac{5}{21} + (5)\frac{8}{21} = 0$$

and

$$y = (0)\alpha + (12)\beta + (0)\gamma = (0)\frac{8}{21} + (12)\frac{5}{21} + (0)\frac{8}{21} = \frac{20}{7}.$$

We see that $(x,y) = (0, 20/7)$.

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Example 2. Let $\triangle ABC$ be the right triangle in the xy-plane with A, B and C at $(0,0)$, $(0,16)$ and $(12,0)$, respectively. Let Z again represent the point at which the cevians to the half-way points are concurrent. Let us find the Cartesian coordinates of Z.

Solution. Since $a = 20$, $b = 12$ and $c = 16$, the convex coordinates of Z as given by equation 4 are $(6/11, 2/11, 3/11)$. The Cartesian coordinates of Z are $x = 0\left(\frac{6}{11}\right) + 0\left(\frac{2}{11}\right) + 12\left(\frac{3}{11}\right) = \frac{36}{11}$ and $y = 0\left(\frac{6}{11}\right) + 16\left(\frac{2}{11}\right) + 0\left(\frac{3}{11}\right) = \frac{32}{11}$.

Another Three Cevians. We close with a second triple of concurrent cevians that may with interest be compared with the triple of cevians to the half-way points. We again consider $\triangle ABC$ with sides of length a, b and c as before. We draw three circles, one centered at each vertex of the triangle so that each circle is externally tangent to the other two as shown in Figure 3.

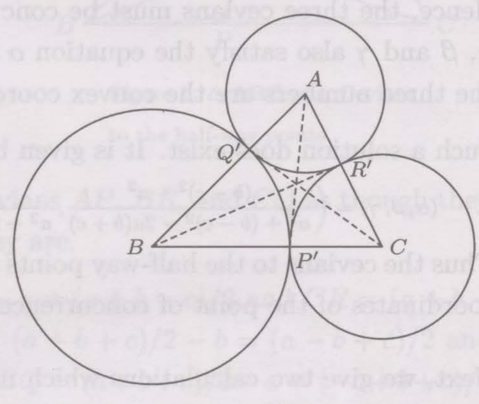

Figure 3. $\triangle ABC$ with Circles at the Vertices

The points at which the circles are tangent in pairs fall on the sides of the triangle. Points of tangency P', R', abd Q' are on the sides opposite to A, B, abd C, respectively. Let the lengths of the radii of the circles centered at A, B, and C be u, v and w respectively. Then

$$\begin{array}{rcl} v + w &=& a, \\ u \quad + w &=& b, \text{ and} \\ u + v \quad &=& c. \end{array} \tag{5}$$

We solve equations 5 simultaneously to find that

$$\begin{array}{rcl} u &=& (-a+b+c)/2, \\ v &=& (a-b+c)/2, \text{ and} \\ w &=& (a+b-c)/2. \end{array} \tag{6}$$

We have drawn cevians AP', BR' and CQ' to be concurrent. That we have drawn the cevians correctly follows immediately from Ceva's Theorem since

$$(AR'/R'C)(CP'/P'B)(BQ'/Q'A) = (u/w)(w/v)(v/u) = 1.$$

It should be clear from Figure 3 that $AR' = Q'A = u$, $BQ' = P'B = v$, and $CP' = R'C = w$.

It strikes us as most interesting that, in both cases, the cevians to the half-way points and the cevians to the points of tangency divide the sides of $\triangle ABC$ into segments which have (in pairs) the same lengths. However, the orders of the lengths

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of the segments around the triangle are different for the two sets of cevians.

We see that for the half-way points $AR = BP = ((a+b-c)/2$, $CP = AQ = (a-b+c)/2$, and $CR = BQ = (-a+b+c)/2$ while for the points of tangency $CP' = CR' = (a+b-c)/2$, $BQ' = BP' = (a-b+c)/2$, and $AR' = AQ' = (-a+b+c)/2$.

We can find the convex coordinates of the point at which the cevians to the points of tangency are concurrent by solving the equations.

$$\begin{aligned} w\alpha' &= u\gamma', \\ u\beta' &= v\alpha', \text{ and} \\ \alpha'+\beta'+\gamma' &= 1. \end{aligned} \tag{7}$$

In equations 7, α', β' and γ' are the convex coordinates with respect to A, B and C of the point at which the cevians to the points of tangency are concurrent. We find that

$$(\alpha', \beta', \gamma') = \left(\frac{u}{u+v+w}, \frac{v}{u+v+w}, \frac{w}{u+v+w}\right).$$

Recalling the solutions for u, v and w given by equations 6, we can also write

$$\alpha' = \frac{-a+b+c}{a+b+c}, \ \beta' = \frac{a-b+c}{a+b+c}, \text{ and } \gamma' = \frac{a+b+c}{a+b+c}.$$

It seems to us that these convex coordinates have a particularly pleasing form. The differences in the computations for the two triples of cevians follow from the differing orders of the segment lengths around $\triangle ABC$. That order matters is a most important lesson.

Example 3. Let $\triangle ABC$ be the right triangle described in Example 2. Let us find the Cartesian coordinates of the point at which the cevians to the points of tangency are concurrent.

Solution. Since $a = 20$, $b = 12$ and $c = 16$, $(\alpha', \beta', \gamma') = (1/6, 1/2, 1/3)$. Therefore, the Cartesian coordinates (x, y) are given by

$$x = 0\left(\frac{1}{6}\right) + 0\left(\frac{1}{2}\right) + 12\left(\frac{1}{3}\right) = 4 \text{ and } y = 0\left(\frac{1}{6}\right) + 16\left(\frac{1}{2}\right) + 0\left(\frac{1}{3}\right) = 8.$$

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