Competition Corner

Please send your solutions and all other communications about this column to

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1. (Russian Mathematical Olympiad, 2003) Point K is chosen on the diagonal AC of the convex quadrilateral $ABCD$ so that $KD = DC$, $\angle BAC = \frac{1}{2}\angle KDC$, $\angle DAC = \frac{1}{2}\angle KBC$. Prove that either $\angle KDA = \angle BCA$ or $\angle KDA = \angle KBA$.

2. (Czech and Slovak Mathematical Olympiad, 2003) A sequence $(x_n)_{n=1}^\infty$ of integers whose first member $x_1 = 1$ satisfies the condition

$$x_n = \pm x_{n-1} \pm x_{n-2} \cdots \pm x_1$$

for a suitable choice of the signs "+" and "−", for any $n > 1$; for instance, $x_2 = -x_1$, $x_3 = -x_2 + x_1$, $x_4 = x_3 - x_2 - x_1$, ... For a given n, find all possible values of x_n.

3. (Hong Kong Mathematical Olympiad, 2003) Let $n \geq 3$ be an integer. In a conference there are n mathematicians. Every pair of mathematicians communicate in one of n official languages of the conference. For any three different official languages, there exist three mathematicians who communicate with each other in these three languages. Determine all n for which this is possible. Justify your claim.

4. (Romanian Mathematical Olympiad, 2003) An integer n, $n \geq 2$ is called *friendly* if there exists a family $A_1, A_2, \ldots, A_n$ of subsets of the set $\{1, 2, \ldots, n\}$ such that:

(1) $i \notin A_i$ for every $i \in \{1, 2, \ldots, n\}$;

(2) $i \in A_j$ if and only if $j \notin A_i$, for every distinct $i, j \in \{1, 2, \ldots, n\}$;

(3) $A_i \cap A_j$ is nonempty, for every $i, j \in \{1, 2, \ldots, n\}$.

Prove that n is friendly if and only if $n \geq 7$.

5. (Russian Mathematical Olympiad, 2003) Let $f(x)$ and $g(x)$ be polynomials with nonnegative integer coefficients and that m is the largest coefficient of f. It is known that for some natural numbers $a < b$, the equalities $f(a) = g(a)$ and $f(b) = g(b)$ are true. Prove that if $b > m$, then f and g are identical.

6. (Austrian Mathematical Olympiad, 2002) Let $ABCD$ and $AEFG$ be similar cyclic quadrilaterals, whose vertices are labeled counterclockwise. Let P be the second common point of the circumcircles of the quadrilaterals beside A, show that P must lie on the line connecting B and E.

7. (Belarusian Mathematical Olympiad, 2003) Given a convex pentagon $ABCDE$ with $AB = BC$, $CD = DE$, $\angle ABC = 150°$, $\angle CDE = 30°$, $BD = 2$, find the area of $ABCDE$.

8. (Korean Mathematical Olympiad, 2003) Show that there exist no integers x, y, z satisfying

$$2x^4 + 2x^2y^2 + y^4 = z^2, \quad x \neq 0.$$

9. (Iranian Mathematical Olympiad, 2003) Find the smallest positive integer n such that:

"For any finite set of points in the plane, if for every n points of this set, there exist two lines covering all n points, then there exist two lines covering all the set."

10. (Finland Mathematical Olympiad, 2003)

Players Aino and Eino take turns in choosing numbers from the set $\{0, 1, \ldots, n\}$, where $n \in \mathbb{N}$ is a predetermined number. The game ends when the numbers chosen by either player contain a set of four numbers that can be arranged to an arithmetic progression. The player whose numbers contain such progression wins the game. Show that there exists an n for which the first player has a winning strategy. Find the least possible n for which this is possible.

1. (IMO 2003 shortlisted problems.) Three distinct points A, B, C are fixed on a line in this order. Let Γ be a circle passing A and C whose centre does not lie on the line AC. Denote by P the intersection of the tangents to Γ at A and C. Suppose Γ meets the segment PB at Q. Prove that the bisector of $\angle AQC$ and the line AC intersect at a point which does not depend on the choice of Γ.

Solution by Joel Tay Wei En. Let the point be R, $AQ = x$, $QC = y$. By the angle bisector theorem, $AR/RC = x/y$. Let $\angle PAQ = s$, $\angle PCQ = t$, $\angle APB = u$, $\angle CPB = v$. Then $s = \angle PAQ = \angle QCA$ and $t = \angle PCQ = \angle QAC$. By using the sine rule we get

$$\frac{x}{\sin s} = \frac{y}{\sin t}, \qquad (\triangle ACQ)$$

$$\frac{AB}{\sin u} = \frac{BC}{\sin v}, \qquad (\triangle ABP, \triangle BCP)$$

$$\frac{x}{\sin u} = \frac{PQ}{\sin s}, \qquad (\triangle APQ)$$

$$\frac{y}{\sin v} = \frac{PQ}{\sin t}. \qquad (\triangle CPQ)$$

These imply that

$$\frac{x}{y} = \sqrt{\frac{\sin u}{\sin v}} = \sqrt{\frac{AB}{BC}}$$

which is independent of Γ.

Also solved by Kenneth Tay Jingyi, Anand B. Rajagopalan, Andre Kueh Ju Lui, Ong Xing Congand A. Robert Pargeter.

2. (Canada Mathematical Olympiad, 2003) Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart. Prove there is a subset T of S with at least $n/7$ points such that any two points of T are at least $\sqrt{3}$ units apart.

Similar solution by Joel Tay Wei En and Andre Kueh Ju Lui. Let X be a point on the boundary of the convex hull of S. We shall show that there is a set S_x with most 6 points such that their distances from X are in the range $[1, \sqrt{3})$. Then we can put X in T and consider the set S' obtained by deleting X and S_x. By continuing this process, we can construct a set T with at least $n/7$ points.
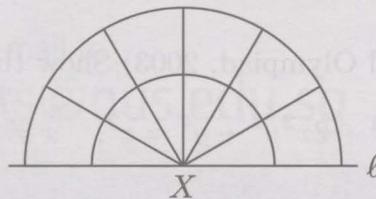
Now let ℓ be the line containing X such that all the points in S lie on one side of ℓ. With X as the centre, draw two semicircles with radii 1 and $\sqrt{3}$. Divide the semicircles into 6 equal sectors. Since in each of the 6 regions in the shaded area, any two points are at most 1 unit apart, the shaded region contains at most 6 points of S.

3. (German National Mathematical Competition, 1st round, 2003.) Determine, with proof, the set of all positive integers that cannot be represented in the form $\frac{a}{b} + \frac{a+1}{b+1}$, where a and b are positive integers.

Similar solution by Joel Tay Wei En, Andre Kueh Ju Lui, Anand B. Rajagopalan, Ong Xing Cong, Kenneth Tay Jingyi, Charmaine Sia Jia Min and Daniel Chen Chongli.
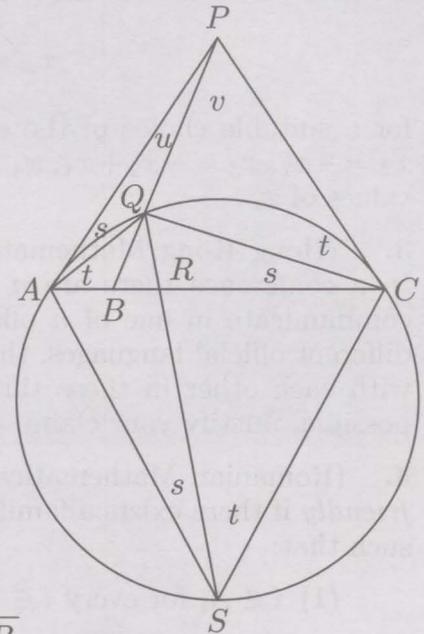
Let A denote the given expression. Then $A = \frac{2ab+a+b}{b(b+1)}$ implies $b \mid a$. Let $a = mb$, $m \in \mathbb{N}$. Then $A = m + \frac{mb+1}{b+1} = 2m - \frac{m-1}{b+1}$ implies $b + 1 \mid m - 1$. Let $m - 1 = n(b + 1)$, $n \geq 0$, $n \in \mathbb{Z}$. Then $A = n(2b + 1) + 2$. If $n = 0$, $A = 2$. If $n = 1$, then by varying b, we get $A = 5, 7, \ldots$. We also note that $A \neq 0, 3$. We are left with even numbers $x > 2$. $A = x$ if and only if $x - 2 = n(2b + 1)$ if and only if $x - 2$ is a multiple of some odd prime. Thus the required set is $\{1, 2^k + 2 : k \in \mathbb{N}\}$.

4. (Hong Kong Mathematical Olympiad, 2003) Let p be an odd prime such that $p \equiv 1 \pmod 4$. Evaluate, with reasons,

$$\sum_{k=1}^{\frac{p-1}{2}} \left\{ \frac{k^2}{p} \right\},$$

where $\{x\} = x - \lfloor x \rfloor$, $\lfloor x \rfloor$ being the greatest integer not exceeding x.

Similar solution by Kenneth Tay Jingyi and Andre Kueh Ju Lui. Since $k^2 \equiv (p - k)^2 \pmod p$, the set $S = \{k^2 \mid k = 1, 2, \ldots, m\}$, where $m = (p - 1)/2$, is the entire set of quadratic residues $\pmod p$. Also if $1 \leq a \leq b \leq m$ satisfies $a^2 \equiv b^2 \pmod p$, we get $(a + b)(a - b) \equiv 0 \pmod p$. But $a + b \leq 2m = p - 1$, therefore $a = b$. Thus the elements k^2, $k = 1, 2, \ldots, m$, are distinct.

Moreover, by Wilson's theorem, $(p - 1)! \equiv -1 \pmod p$. Since $i(p - i) \equiv -i^2 \pmod p$, we see that $(-1)^m (m!)^2 \equiv -1 \pmod p$. Since m is even, we have $(m!)^2 \equiv -1 \pmod p$. Thus if $x^2 \equiv i$, then $(m!x)^2 \equiv -i \pmod p$. So if $i \in S$, then so is $p - i$. Therefore the answer is $(p - 1)/4$.

5. (British Mathematical Olympiad, 2003) Let $f : \mathbb{N} \to \mathbb{N}$ be a permutation of the set $\mathbb{N}$ of all positive integers.

    (i) Show that there is an arithmetic progression of positive integers $a, a + d, a + 2d$, where $d > 0$, such that

$$f(a) < f(a + d) < f(a + 2d).$$

    (ii) Must there be an arithmetic progression $a, a + d, \ldots, a + 2003d$, where $d > 0$ such that

$$f(a) < f(a + d) < \cdots < f(a + 2003d)?$$

Similar solution by Kenneth Tay Jingyi, Joel Tay Wei En, Charmaine Sia Jia Min and Andre Kueh Ju Lui.

    (i) Assume on the contrary that no such AP exists. Let m be the positive integer such that $f(m) = 1$. Then $f(m+1) > f(m+2)$ otherwise $m, m+1, m+2$ is such an AP. Similarly, $f(m+2) > f(m+4) > f(m+8) > \cdots$. But this means there is an infinite sequence of strictly decreasing positive integers. This gives rise to a contradiction. So such an AP must exist.

    (ii) Consider the following permutation where $f([2^k, 2^{k+1})) = [2^k, 2^{k+1})$ and f is decreasing in the interval. Thus

$$f(1) = 1; \quad f(2) = 3, f(3) = 2; \quad f(4) = 7, f(5) = 6, f(6) = 5, f(7) = 8; \quad \text{etc.}$$

If such an AP exists, then each of the following intervals contain at most one term of the AP:

$$[1, 2), [2, 4), [4, 8), \ldots, [2^k, 2^{k+1}), \quad \text{where} \quad a + 2003d \in [2^k, 2^{k+1}).$$

Since $a + 2001d \leq 2^{k-1} - 1$, we see that $d \geq 2^{k-2} + 1$. Thus $a + 2003d \geq 2^{k+1}$, a contradiction.

6. (Vietnam Mathematical Olympiad, 2003) Two circles Γ_1 and Γ_2 with centres O_1 and O_2, respectively, touch each other at the point M. The radius of Γ_2 is larger that of Γ_1. A is a point on Γ_2 such that the points O_1, O_2 and A are not collinear. Let AB and AC be tangents of Γ_1 with touching points B and C. The lines MB and MC meet Γ_2 again at E and F, respectively. Let D be the point of intersection of the line EF and the tangent to Γ_2 at A. Prove that the locus of D is a straight line when A moves on Γ_2 so that O_1, O_2 and A are not collinear.

Solution by Calvin Lin. Let NE and NA be tangents to Γ_2. M is the centre of the homothety that takes Γ_1 o Γ_2. Thus the tangent of Γ_1 at B is parallel to the tangent of Γ_2 at E, i.e., $NE \parallel AB$. Thus $\angle BAE = \angle AEN = \angle EMA$, and whence NA is tangent to the circumcircle of $\triangle BAM$. Thus, by considering the power of E with respect to the circumcircle, $EA^2 = EB.EM$. Thus the powers of E with respect to Γ_1 and Γ_3, the point circle A, are equal. The same goes for F, i.e., EF is the radical axis of Γ_1 and Γ_3. Since the radical axes of Γ_1 and Γ_2 is the tangent at M and that of Γ_2 and Γ_3 is the tangent at A, we see that D lies on the tangent at M.
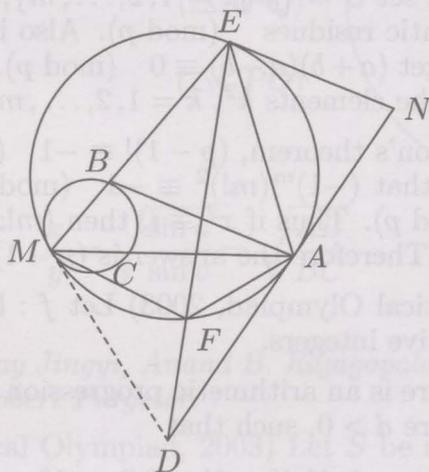

Conversely, let X be any point on the tangent of Γ_2 at M. Let XA be the other tangent. With B, C, E, F as defined, the above argument shows that E, F, X are collinear. Thus every point on the tangent at M satisfies the required properties. Thus the locus is the tangent at M.

The same proof works when Γ_1 and Γ_2 are externally tangent to each other.

7. (Hong Kong Mathematical Olympiad, 2003) Two circles intersect at points A and B. Through the point B a straight line is drawn, intersecting the first circle at K and the second circle at M. A line parallel to AM is tangent to the first circle at Q. The line AQ intersects the second circle again at R.

(a) Prove that the tangent to the second circle at R is parallel to AK.

(b) Prove that these two tangents are concurrent with KM.
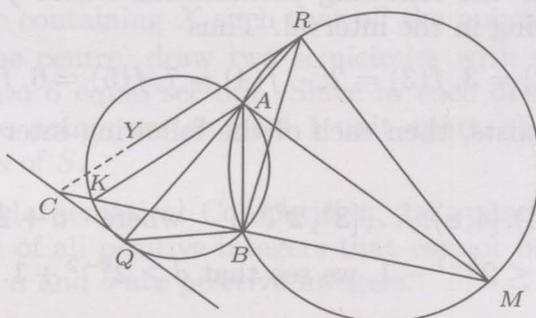
Solution by Anand B. Rajagopalan. (a) Let RY be the tangent at R and C be the intersection of KM with the tangent at Q. By the alternate segment theorem and the fact that $CQ \parallel AM$, we have $\angle QBA = \angle CQA = \angle QAM$. But $\angle QAM = 180° - \angle MAR = 180° - \angle MBR = \angle KBR$. Therefore $\angle QBA = \angle KBR$ and $\angle QBK = \angle ABR$. But $\angle QBK = \angle QAK$ and $\angle ABR = \angle ARY = \angle QRY$. Therefore $\angle QAK = \angle QRY$ whence $AK \parallel RY$.

(b) We have

$$\angle QCB = \angle BMA = \angle BRA = \angle QRB$$

and therefore Q, B, R, C are concyclic. Let RY intersect MK at D. Then

$$\angle RDB = \angle AKB = \angle AQB = \angle RQB$$

and therefore R, Q, Q, B are concyclic. Since C and D both lie on MK, they are the same point and we are done.

Also solved by Andre Kueh Ju Lui, Kenneth Tay Jingyi and Daniel Chen Chongli.

8. (Belarussian Mathematical Olympiad, 2003) Let

$$f(x) = (x+1)^p (x-3)^q = x^n + a_1 x^{n-1} + a_2 x_{n-2} + \cdots + a_{n-1} x + a_n,$$

where p and q are positive integers.

(a) Given that $a_1 = a_2$, prove that $3n$ is a perfect square.

(b) Prove that there exist infinitely many pairs (p, q) of positive integers p and q such that the equality $a_1 = a_2$ is valid for the polynomial $p(x)$.

Solution by A. Robert Pargeter. (a) Since

$$(x+1)^p = x^p + px^{p-1} + \frac{p(p-1)}{2}x^{p-2} + \cdots,$$

$$(x-3)^q = x^q - 3qx^{q-1} + \frac{9q(q-1)}{2}x^{q-2} + \cdots,$$

we have $n = p + q$, $a_1 = p - 3q$, $a_2 = \frac{9q^2 - 9q + p^2 - p - 6pq}{2}$. Therefore

$$a_1 = a_2 \quad \Leftrightarrow \quad 2p - 6q = 9q^2 - 9q + p^2 - p - 6pq \quad \Leftrightarrow \quad (3q - p)^2 = 3(p + q).$$

Since $n = p + q$, we are done.

(b) Let $p = 36t^2 \mp 3t$, $q = 12t^2 \pm 3t$, where $t \in \mathbb{N}$. Then

$$(3q - p)^2 = 144t^2 = 3p + 3q.$$

Thus $a_1 = a_2$. Since this is true for any t, the proof is complete.

Also solved by Andre Kueh Ju Lui, Joel Tay Wei En, Kenneth Tay Jingyi, Daniel Chen Chongli and Anand B. Rajagopalan.

9. (Russia Mathematical Olympiad, 2003) Find the greatest natural number N such that for any arrangement of the natural numbers $1, 2, \ldots, 400$ in the cells of a 20×20 square table there exist two numbers located in the same row or in the same column such that their difference is not less than N.

Similar solution by Kenneth Tay Jingyi and Andre Kueh Ju Lui. The following array shows that $N \leq 209$:

1	2	$\cdots$	10	201	202	$\cdots$	210
11	12	$\cdots$	20	211	212	$\cdots$	220
$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$		$\vdots$
191	192	$\cdots$	200	391	392	$\cdots$	400

Next we shall prove that $N \geq 209$ which would then give the conclusion that $N = 209$. Consider the numbers $1, 2, \ldots, 91$. Suppose they occupy n rows and m columns. Then $mn \geq 91$, whence $m + n \geq 2\sqrt{mn} > 19$. Thus $n + m \geq 20$. Now suppose that the numbers $300, 301, \ldots, 400$ occupy i rows and j columns. Then similar consideration shows that $i + j \geq 21$. Thus there is a row or a column that contains a number from $\{1, 2, \ldots, 91\}$ and a number from $\{300, 301, \ldots, 400\}$. This shows $N \geq 209$ and we are done.

10. (Czech and Slovak Mathematical Olympiad, 2003) Find all possible values of the expression

$$\frac{a^4 + b^4 + c^4}{a^2b^2 + a^2c^2 + b^2c^2},$$

where a, b, c are the lengths of the sides of a triangle.

Solution by Joel Tay Wei En.

Let X denote the given expression. Since $(a^2 - b^2)^2 + (b^2 - c^2)^2 + (c^2 - a^2)^2 \geq 0$, we have $X \geq 1$, with equality if and only if $a = b = c$.

By cosine rule, we have $2bc > |b^2 + c^2 - a^2|$, since $\cos A \neq 1$. Similarly, we have $2ab > |a^2 + b^2 - c^2|$, and $2ac > |a^2 + c^2 - b^2|$. Square both sides in each expression and sum, we get

$$2(b^2c^2 + c^2a^2 + a^2b^2) > a^4 + b^4 + c^4.$$

Hence $X < 2$ and therefore $X \in [1, 2)$. To see that X assumes every value $k \in [1, 2)$, in the interval, we let $a = b = 1$ and $c = y^2$. This yields the quadratic equation $p(y) = y^2 - 2ky + (2 - k) = 0$. Since $p(1) \leq 0$ and $p(y)$ is positive for large y, this equation has a solution. Thus there exists c such that

$$\frac{1^4 + 1^4 + c^4}{1^2 1^2 + 1^2 c^2 + 1^2 c^2} = k.$$

If we allow degenerate triangle, then the range is extended to $[1, 2]$.

Also solved by A. Robert Pargeter.