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An Insight Into The Link Between Disk Method and Shell Method

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In the topic of finding the volume of the solid of revolution generated by revolving a region in the Cartesian Plane about an axis, one may come across integral such as $\pi \int x^2 dy$, formulated by the well-known disk-method. As y is usually given as a function of x, evaluation of such integral could be a complicated task, particularly so if x and y are not one-to-one related. To overcome the difficulties, shell-method is available as an alternative. The integral then takes the form $2\pi \int xy dx$, thereby avoiding the work of putting x in terms of y.

As the two methods are based on different ways of taking elements of volume, they are usually perceived as two different tools, of which, the better one has to be chosen at the start of the work. We shall see their connection, and show that each is in fact derivable from the other mathematically.

For definiteness we shall restrict our discussion to the volume of the solid of revolution generated by revolving a region in the first quadrant of the Cartesian Plane about the y-axis. In this article, all arcs are assumed to be continuous; derivatives, integrals, whenever there arises, are all assumed to exist, unless otherwise stated. We first consider typical regions.

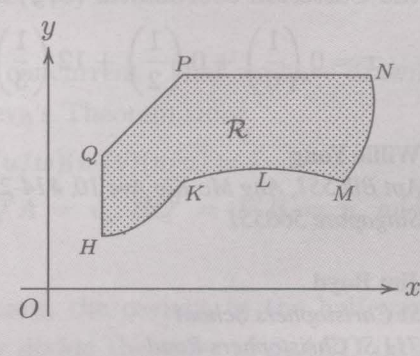
Type I - region whose boundary is oriented in the anticlockwise sense.

Let there be an arc AB in the first quadrant with end points $A(x_A, y_A)$ and $B(x_B, y_B)$ with cases:

(i) the path along the arc AB in the direction from A to B is strictly ascending, and that all values of x and y on the arc are one-to-one (see diagram, HK being an example).

(ii) the arc AB is a straight line parallel to the x-axis (like PN in the diagram)

(iii) the arc AB is a straight line parallel to the y-axis with $y_B > y_A$ (like QH in the diagram).



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Let $\mathcal{R}(AB, y)$ be the region bounded by arc AB, the line $y = y_B$, the y-axis and the line $y = y_A$. We note that $\mathcal{R}(AB, y)$ is described in an anticlockwise sense.

Let $V(\mathcal{R})$ be the volume of the solid of revolution generated by revolving a region $\mathcal{R}$ about the y-axis, and $I_d(\mathcal{R})$, $I_s(\mathcal{R})$ represent the integrals for its evaluation by way of the disk-method and shell-method respectively. For the evaluation of $V(\mathcal{R}(AB, y))$, we have

$$\begin{aligned} I_d(\mathcal{R}(AB,y)) &= \pi \int_{y_A}^{y_B} x^2 dy, \text{ and} \\ I_s(\mathcal{R}(AB,y)) &= 2\pi \int_0^{x_A} x(y_B - y_A)\, dx + 2\pi \int_{x_A}^{x_B} x(y_B - y)\, dx, \\ &= 2\pi \int_0^{x_B} x(y_B - y_A) dx + 2\pi \int_{x_B}^{x_A} x(y - y_A)\, dx, \\ &= \pi(x_B^2 y_B - x_A^2 y_A) - 2\pi \int_{x_A}^{x_B} xy dx. \end{aligned}$$

We shall prove that $I_d(\mathcal{R}(AB, y)) = I_s(\mathcal{R}(AB, y))$ for the three cases.

(i) $I_d(\mathcal{R}(AB,y)) = \pi \int_{y_A}^{y_B} x^2\, dy = \pi \int_{x_A}^{x_B} x^2 \frac{dy}{dx} dx = \pi [x^2 y]_{x_A}^{x_B} - 2\pi \int_{x_A}^{x_B} xy\, dx.$

Using integration by parts and $\int \frac{dy}{dx} dx = y + C$, the above expression equals to

$$\pi(x_B^2 y_B - x_A^2 y_A) - 2\pi \int_{x_A}^{x_B} xy\, dx = I_s(\mathcal{R}(AB, y)).$$

(ii) $I_d(\mathcal{R}(AB,y)) = I_s(\mathcal{R}(AB,y)) = 0$.

(iii) $I_d(\mathcal{R}(AB,y)) = \pi \int_{y_A}^{y_B} x^2 dy = \pi x_A^2 (y_B - y_A) = \pi x_B^2 (y_B - y_A) = \pi(x_B^2 y_B - x_A^2 y_A)$. As $x_A = x_B$, we may write it as $\pi(x_B^2 y_B - x_A^2 y_A) - 2\pi \int_{x_A}^{x_B} xy\, dx = I_s(\mathcal{R}(AB, y))$, where the value of the second integral is zero.

Type II - region whose boundary is oriented in the clockwise sense.

Consider now arc BA, with arc AB as described above.

Let $\mathcal{R}(BA, y)$ be the region which is bounded by arc BA, the line $y = y_A$, the y-axis and the line $y = y_B$. We note that $\mathcal{R}(BA, y)$ is described in an clockwise sense.

We see that the volume of the solid of revolution generated by revolving $\mathcal{R}(BA, y)$ about the y-axis has its absolute value equals $V(\mathcal{R}(AB, y))$. Also

$$\pi \int_{y_B}^{y_A} x^2 dy = -\pi \int_{x_A}^{y_B} x^2 dy$$

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$$\pi(x_A^2 y_A - x_B^2 y_B) - 2\pi \int_{x_B}^{x_A} xydx = -\left[\pi(x_B^2 y_B - x_A^2 y_A) - 2\pi \int_{x_A}^{x_B} xydx\right].$$

For convenience, we define

$$V(\mathcal{R}(BA, y)) = -V(\mathcal{R}(AB, y)),$$

$$I_d(\mathcal{R}(BA, y)) = -I_d(\mathcal{R}(AB, y))$$

and

$$I_s(\mathcal{R}(BA, y)) = -I_s(\mathcal{R}(AB, y)),$$

so

$$V(\mathcal{R}(BA, y)) = -V(\mathcal{R}(AB, y)) = -I_d(\mathcal{R}(AB, y)) = -I_s(\mathcal{R}(AB, y)).$$

This last statement can also be a general statement now, as long as the path along the arc AB is either non-descending or non-ascending.

We now look into any region in the first quadrant.

Let a region be $\mathcal{R}(A_0A_1 \ldots A_{n-1}A_0)$, bounded by a closed but non-self-intersecting curve $A_0A_1A_2 \ldots A_{n-1}A_0$, described in an anticlockwise sense, A_0 being the lowest point, in such a way that, $A_0A_1, A_1A_2, \ldots A_{n-1}A_0$ are arcs either of the type I or type II as discussed above. (See diagram, $\mathcal{R}(HKLMNPQH)$ serves as an example)

Taking A_n to be A_0, if the arc A_iA_{i+1} is of the type I, $\mathcal{R}(A_iA_{i+1}, y)$ is a sweeping-in region for $\mathcal{R}(A_0A_1 \ldots A_{n-1}A_0)$ and $V(\mathcal{R}(A_iA_{i+1}, y))$ is positive.

If the arc A_jA_{j+1} is of type II, then $\mathcal{R}(A_jA_{j+1}, y)$ is a sweeping-off region for $\mathcal{R}(A_0A_1 \ldots A_{n-1}A_0)$ and $V(\mathcal{R}(A_jA_{j+1}, y)$ is negative.

Thus, for $V(\mathcal{R}(A_0A_1 \ldots A_{n-1}A_0))$.

$$\begin{aligned} I_d(\mathcal{R}(A_0A_1 \ldots A_{n-1}A_0)) &= \sum_{i=0}^{n-1} I_d(R(A_iA_{i+1}, y)) \\ &= \sum_{i=0}^{n-1} I_s(R(A_iA_{i+1}, y)) \\ &= I_s(\mathcal{R}(A_0A_1 \ldots A_{n-1}A_0)). \end{aligned}$$

Thus the two integrals are derivable from each other.

Applying some skill employed in the above discussion, perhaps, we might say that disk-method is quite sufficient for finding volume of solid of revolution of a region bounded by straight lines and continuous arcs, classified as above, if differentiation, integration are all possible whenever it requires.

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Facing an integral such as $\pi \int_{y_A}^{y_B} x^2\, dy$, we can proceed in the following way:

$$\pi \int_{y_A}^{y_B} x^2 dy = \pi \int_{x_A}^{x_B} \left(x^2 \frac{dy}{dx} \right) dx, \quad \text{(just be sure that } x \text{ and } y \text{ are one-to-one).}$$

The integration with respect to x may prove to be easy.

If not, we can apply integration by parts:

$$\pi \int_{x_A}^{y_B} x^2 dy = \pi \int_{x_A}^{x_B} (x^2) \left(\frac{dy}{dx} dx \right) = \pi \left[x^2 y - \int 2xy dx \right]_{x_A}^{x_B}.$$

Shell-method is actually embedded in this process.

For a less well-conditioned case where the arcs and straight lines are just continuous, as such obtained by usual act of drawing, yet existence of derivatives inside the integrals not assured as when we need, we can still show that disk-method and shell-method are derivable from each other by using the Riemann sums.

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