Mathematical Medley Problems Corner

A Prized Problems

Problem 1

(Book voucher up to $150)

A cubic block is partitioned into n^3 unit cubic blocks. Two unit blocks are adjacent if they share a common face.

Two players *A* and *B* play a game as follow:

A starts from any unit block and moves to an adjacent unit block. Then *B* moves from this new block to another adjacent block. The two players move alternately without revisiting any unit block. The first player who cannot move on lose the game.

Suppose both players are playing with their best strategies. Who will win the game?

Problem 2

(Book voucher up to $150)

Let A be a set of k natural numbers.
Denote by $A+A$ the set $\{n : n = a + b \text{ where } a, b \in A\}$.
Show that $A+A$ has $2k - 1$ (distinct) elements if and only if
A contains an arithmetic progression of length k.

B Instruction

1. Prizes in the form of book vouchers will be awarded to one or more received best solutions submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

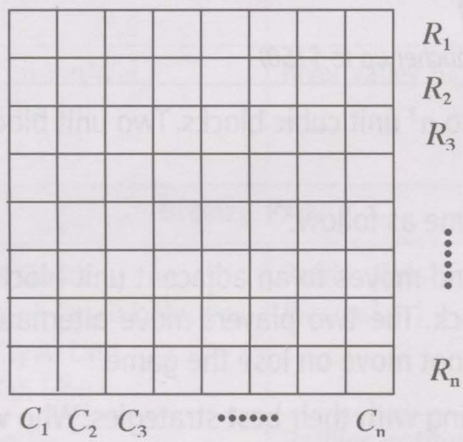
3. Solutions should be sent to : The Editor, Mathematical Medley, c/o Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore117543 ; and should arrive before 1 October 2004.
   Alternatively, softcopies of the solutions can also be sent to the email address: mattanv@nus.edu.sg.

4. The Editor's decision will be final and no correspondence will be entertained.

Solutions to the problems of volume 30, No2, 2003

Problem 1

Consider an $n \times n$ array with entries either 1 or -1. Let R_i be the product of the entries in row i and C_i be the product of the entries in column i $(1 \le i \le n)$.



How many ways are there to arrange 1 and -1 in the boxes so that $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i$?

Solution 1

By Charmaine Sia Jia Min - Raffles Junior College

Consider an $n \times n$ array with the $(n-1) \times (n-1)$ array in the upper left hand corner already filled with entries either 1 or -1. Let x_{ij} be the entry in row *i*, column *j*.

We see that each R_i (respectively C_i) is uniquely determined by x_{ni} (respectively x_{in}). Let

$$r_i = \prod_{j=1}^{n-1} x_{ij} \text{ and } c_i = \prod_{j=1}^{n-1} x_{ji}.$$

We will split the ways of arranging 1 and -1 in the boxes so that $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i$ into 2 cases. In both cases we first show that the resulting arrangement fulfils the given conditions.

Case 1: $\sum_{i=1}^{n-1} R_i = \sum_{i=1}^{n-1} C_i.$

Let the desired number of values of $R_i = 1$ among the first $n-1$ rows (i.e. $1 \le i \le n-1$) be k. There are $\binom{n-1}{k}$ ways of selecting k rows such that $R_i = 1$ for those k rows and $R_i = -1$ for the remaining $n-1-k$ rows. For the k selected rows, let the final entry in that row be r_i and for the remaining $n-1-k$ rows, let the final entry in that row be $-r_i$. It is easy to see that the number of values of $C_i = 1$ among the first $n-1$ columns is k. Apply a similar procedure for the columns.

Now observe that $\prod_{i=1}^{n-1} r_i = \prod_{i=1}^{n-1} c_i$ because it is the product of all the entries in the $(n-1)\times(n-1)$ array in the upper left hand corner, and $\prod_{i=1}^{n-1} R_i = \prod_{i=1}^{n-1} C_i$ by construction.

$\therefore\ r_n = c_n \rightarrow R_n = C_n$. Thus the given conditions of the question are satisfied.

Total number of ways of filling in last row and last column

$$= \sum_{k=1}^{n-1} 2\binom{n-1}{k}\binom{n-1}{k} \quad \text{since } x_{nn} \text{ can be either 1 or } -1$$

$$= 2\binom{2n-2}{n-1} \quad \text{by Vandermonde's identity.}$$

Case 2: $\sum_{i=1}^{n-1} R_i \neq \sum_{i=1}^{n-1} C_i$.

Since $R_n = 1$ or -1, $C_n = 1$ or -1, to have $\sum_{i=1}^{n} R_i = \sum_{i=1}^{n} C_i$, we must either have $\sum_{i=1}^{n-1} R_i = \sum_{i=1}^{n-1} C_i + 2$ or $\sum_{i=1}^{n-1} R_i + 2 = \sum_{i=1}^{n-1} C_i$.

WLOG we consider $\sum_{i=1}^{n-1} R_i = \sum_{i=1}^{n-1} C_i + 2$. This means that if there are k values of R_i such that $R_i = 1$ among the first $n-1$ rows, then there are $k-1$ values of C_i such that $C_i = 1$ among the first $n-1$ columns.

We select k rows where the final entry in that row is r_i and the final entry for the remaining $n-1-k$ rows is $-r_i$.

We also select $k-1$ columns where the final entry in that column is c_i and the final entry for the remaining $n-k$ columns is $-r_i$. Since $\prod_{i=1}^{n-1} r_i = \prod_{i=1}^{n-1} c_i$ and $\prod_{i=1}^{n-1} R_i = -\prod_{i=1}^{n-1} C_i \rightarrow r_n = -c_n$. We select x_{nn} such that $R_n = -1$ and $C_n = 1$, thus satisfying the conditions of the question. Observe that there is only one way to select such x_{nn}.

The argument for $\sum_{i=1}^{n-1} R_i + 2 = \sum_{i=1}^{n-1} C_i$ is similar.

Total number of ways of filling in last row and last column

$$= 2\sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n-1}{k-1}$$
$$= 2\sum_{k=1}^{n-1}\binom{n-1}{k}\binom{n-1}{n-k}$$
$$= 2\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n-1}{n-k}$$
$$= 2\binom{2n-2}{n} \text{ by Vandermonde's identity.}$$

Since there are $2^{(n-1)^2}$ ways of filling in the $(n-1)\times(n-1)$ array at the top left hand corner,

total no. of ways of filling in the boxes

$$= 2^{(n-1)^2}\left[2\binom{2n-2}{n-1} + 2\binom{2n-2}{n}\right]$$
$$= 2^{n^2-2n+2}\binom{2n-1}{n}.$$

Solution 2

By Teo Wei Hao - National Junior College

Firstly, we count the number S of possible configurations of $(R_1, R_2, \ldots, R_n, C_1, C_2, \ldots, C_n)$. Suppose k of the R_i, i = 1,2,…,n are -1 and the rest are 1. Then similarly k of the C_i, i = 1,2,…,n are -1 and the rest are 1. Number of ways to place k (-1)'s is $\binom{n}{k}$. Therefore the number of possible configurations is $S = \binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \ldots + \binom{n}{n}^2 = \binom{2n}{n}$

Next we count the number of ways to arrange the array to obtain each configuration of $(R_1, R_2, \ldots, R_n, C_1, C_2, \ldots, C_n)$.

Notice that each configuration is at least obtainable, i.e. to get the configuration such that $R_{i_1}, R_{i_2}, \ldots, R_{i_k}, C_{j_1}, C_{j_2}, \ldots, C_{j_k}$ are -1, just let the entries $(R_{i_1}, C_{j_1}), (R_{i_2}, C_{j_2}), \ldots, (R_{i_k}, C_{j_k})$ in the array be -1 and the remaining entries be 1. We shall call this array our default array for a certain configuration.

Notice that, given any configuration, if we switch the sign of the values in any 2×2 subarray with vertices $(R_p, C_q), (R_p, C_{q+1}), (R_{p+1}, C_{q+1}), (R_{p+1}, C_q)$, the configuration will be preserved. All possible arrays that give the same configuration can be obtained from the default array by a finite set of such transformations. It is not possible to obtain the same array by using a different set of transformations. Since there are $(n-1)^2$ such 2×2 subarrays, we have $2^{(n-1)^2}$ possible arrays for each configuration, which is obtained by choosing whether a 2×2 subarray is being used to transform the default array to our new array.

Therefore the total number of ways is $2^{(n-1)^2}S = 2^{(n-1)^2}\binom{2n}{n}$.

> Editor's note:
> $2\binom{2n-1}{n} = \binom{2n}{n}$. So the two answers given are the same.
> Also solved correctly by Ong Xing Cong (Raffles Junior College).
> The prize is shared by Charmaine, Wei Hao and Xing Cong.

Problem 2

How many solutions are there for the equation $1 = \frac{1}{s_1} + \frac{1}{s_2} + \cdots + \frac{1}{s_8}$ *where s_i are distinct numbers from the arithmetic progression* {2,5,8,11,14,…}?

Solution

By Cong Lin - Hwa Chong Junior College

Firstly, the number of solutions is finite. If not, we will end up using minute numbers whose sum cannot reach 1 at all.

Suppose $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8$ is a solution arranged in increasing order. Let $b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8$ be their reciprocals respectively.

To determine a_1, note that the sum of b_1 and the reciprocals of the 7 numbers immediately following a_1 in the arithmetic sequence is greater than or equal to 1. So a_1 has to be 2.

Now to determine a_2, note that the sum of b_2 and the reciprocals of the 6 numbers immediately following a_2 in the arithmetic sequence is greater than or equal to 1-1/2 = 1/2. So a_2 can only be 5.

To determine a_3, note that the sum of b_3 and the reciprocals of the 5 numbers immediately following a_3 in the sequence is greater than or equal to 1/2 - 1/5 = 3/10. We found out that a_3 can only be 8 or 11.

So now we have two branches. Using similar method in both cases to decide a_4 to a_8, we can get the total number of solutions.
There are also a few useful properties

a. if $x/y = b_5 + b_6 + b_7 + b_8$, then $4y + x$ is divisible by 3
b. if $x/y = b_6 + b_7 + b_8$, then x is divisible by 3.

These properties can help to reduce the number of cases.
I have worked out that there can only be 5 cases from a_1 to a_4. They are:
(2,5,8,11) (2,5,8,14) (2,5,8,17) (2,5,11,14) (2,5,11,17)

The calculation is tremendous, albeit achievable. To be more efficient, I used a Delphi 7 program (available at http://sms.math.nus.edu.sg/program.doc).
The number of solutions generated is 149.

The complete list of solutions is given below:

2 5 8 11 14 80 6161 37951760	2 5 8 11 14 80 7280 40040	2 5 8 11 14 110 296 5180	2 5 8 11 20 35 200 1925
2 5 8 11 14 80 6164 9492560	2 5 8 11 14 80 7370 37520	2 5 8 11 14 110 305 3416	2 5 8 11 20 35 308 440
2 5 8 11 14 80 6167 5426960	2 5 8 11 14 80 7385 37136	2 5 8 11 14 110 308 3080	2 5 8 11 20 38 152 836
2 5 8 11 14 80 6170 3800720	2 5 8 11 14 80 7568 33110	2 5 8 11 14 110 320 2240	2 5 8 11 20 41 104 11726
2 5 8 11 14 80 6176 2377760	2 5 8 11 14 80 7700 30800	2 5 8 11 14 110 329 1880	2 5 8 11 20 41 110 1640
2 5 8 11 14 80 6182 1730960	2 5 8 11 14 80 7760 29876	2 5 8 11 14 110 344 1505	2 5 8 11 20 44 89 7832
2 5 8 11 14 80 6185 1523984	2 5 8 11 14 80 7952 27335	2 5 8 11 14 110 350 1400	2 5 8 11 20 44 92 2024
2 5 8 11 14 80 6188 1361360	2 5 8 11 14 80 8096 25760	2 5 8 11 14 110 380 1064	2 5 8 11 20 44 104 572
2 5 8 11 14 80 6200 954800	2 5 8 11 14 80 8120 25520	2 5 8 11 14 110 392 980	2 5 8 11 20 44 110 440
2 5 8 11 14 80 6209 780560	2 5 8 11 14 80 8360 23408	2 5 8 11 14 110 440 770	2 5 8 11 20 44 152 209
2 5 8 11 14 80 6215 696080	2 5 8 11 14 80 8624 21560	2 5 8 11 14 110 455 728	2 5 8 11 20 50 71 156200
2 5 8 11 14 80 6224 599060	2 5 8 11 14 80 8855 20240	2 5 8 11 14 110 476 680	2 5 8 11 20 50 110 200
2 5 8 11 14 80 6230 548240	2 5 8 11 14 80 8960 19712	2 5 8 11 14 122 224 751520	2 5 8 11 20 56 62 9548
2 5 8 11 14 80 6248 437360	2 5 8 11 14 80 9185 18704	2 5 8 11 14 140 182 40040	2 5 8 11 20 56 77 308
2 5 8 11 14 80 6260 385616	2 5 8 11 14 80 9296 18260	2 5 8 11 14 140 200 1925	2 5 8 11 20 56 110 140
2 5 8 11 14 80 6272 344960	2 5 8 11 14 80 9548 17360	2 5 8 11 14 140 308 440	2 5 8 11 20 65 104 110
2 5 8 11 14 80 6281 319760	2 5 8 11 14 80 9680 16940	2 5 8 11 14 143 182 5720	2 5 8 11 23 26 506 5720
2 5 8 11 14 80 6314 252560	2 5 8 11 14 80 10010 16016	2 5 8 11 14 152 209 770	2 5 8 11 23 32 110 3680
2 5 8 11 14 80 6320 243320	2 5 8 11 14 80 10472 14960	2 5 8 11 17 41 1640 3740	2 5 8 11 23 44 56 35420
2 5 8 11 14 80 6335 222992	2 5 8 11 14 80 10640 14630	2 5 8 11 17 44 395 118184	2 5 8 14 17 23 920 5474
2 5 8 11 14 80 6356 199760	2 5 8 11 14 80 11000 14000	2 5 8 11 17 44 440 3740	2 5 8 14 17 35 62 21080
2 5 8 11 14 80 6380 178640	2 5 8 11 14 80 11060 13904	2 5 8 11 17 44 680 935	2 5 8 14 17 35 68 680
2 5 8 11 14 80 6416 154385	2 5 8 11 14 80 12089 12560	2 5 8 11 17 50 200 3740	2 5 8 14 17 35 80 272
2 5 8 11 14 80 6440 141680	2 5 8 11 14 86 968 728420	2 5 8 11 17 56 140 3740	2 5 8 14 20 23 104 2093
2 5 8 11 14 80 6512 113960	2 5 8 11 14 89 770 7832	2 5 8 11 17 62 110 21080	2 5 8 14 20 26 104 182
2 5 8 11 14 80 6545 104720	2 5 8 11 14 92 560 141680	2 5 8 11 17 65 104 3740	2 5 8 14 20 29 56 812
2 5 8 11 14 80 6560 101024	2 5 8 11 14 92 770 2024	2 5 8 11 17 68 95 28424	2 5 8 14 20 32 56 224
2 5 8 11 14 80 6608 90860	2 5 8 11 14 95 473 45752	2 5 8 11 17 68 110 680	2 5 8 14 20 35 41 1640
2 5 8 11 14 80 6644 84560	2 5 8 11 14 95 476 28424	2 5 8 11 17 80 110 272	2 5 8 14 20 35 44 440
2 5 8 11 14 80 6650 83600	2 5 8 11 14 95 836 1064	2 5 8 11 20 32 353 124256	2 5 8 14 20 35 50 200
2 5 8 11 14 80 6710 75152	2 5 8 11 14 98 407 797720	2 5 8 11 20 32 356 31328	2 5 8 14 20 35 56 140
2 5 8 11 14 80 6776 67760	2 5 8 11 14 98 440 5390	2 5 8 11 20 32 368 8096	2 5 8 14 20 35 65 104
2 5 8 11 14 80 6800 65450	2 5 8 11 14 104 329 134420	2 5 8 11 20 32 374 5984	2 5 8 14 20 44 56 77
2 5 8 11 14 80 6860 60368	2 5 8 11 14 104 572 770	2 5 8 11 20 32 416 2288	2 5 8 14 23 32 35 3680
2 5 8 11 14 80 6944 54560	2 5 8 11 14 110 281 78680	2 5 8 11 20 32 440 1760	2 5 11 14 17 20 35 3740
2 5 8 11 14 80 7007 50960	2 5 8 11 14 110 284 19880	2 5 8 11 20 32 473 1376	
2 5 8 11 14 80 7040 49280	2 5 8 11 14 110 287 11480	2 5 8 11 20 32 608 836	
2 5 8 11 14 80 7238 41360	2 5 8 11 14 110 290 8120	2 5 8 11 20 35 182 40040	

Editor's note:
I only received one solution from Cong Lin. Although there is some limitation in his program, the list of solutions he generated is complete. A book prize of $100 goes to him.