GENERALIZATIONS OF A GEOMETRIC PROBLEM

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ABSTRACT. This paper examines a proof of an Euclidean geometry problem which appears rather straightforward but turns out to be somewhat more challenging than it first appears. One original solution to this problem suggests that it is over-specified as originally stated and that not all given information is required for a complete solution. The implications of this discovery are then discussed in detail and generalized.

1. Introduction

The ideas that contributed to the contents of this paper came initially as much of a surprise to the authors. It started off when one of the authors brought up a problem (Weeks and Adkins [1]; 1961; p. 257) while casually discussing mathematical problem solving among a group of colleagues. It turned out that this 'old' problem generated a great amount of interest among the authors and it seemed that new solutions and perspectives were offered each time discourse over the problem took place over a couple of weeks. This paper is thus a sharing of some of the insights gained as we investigated different approaches to the problem.

On hindsight, as the authors reflect on this short enjoyable journey of joint-problem solving, it was discovered that the value of the enterprise is not merely in the destination, i.e., the solutions, but also in the process taken together. The process involved, among other things, opportunities to verbalise one's 'attack route' to the problem, clarify working steps, offer alternative routes, take time off to mull over the problem, and to seek extensions to the problem. Incidentally, these moves closely mirror the desired practices of project work groups that take an interest in solving mathematics problems. As the schools begin their journey to involve more students in problem solving projects, it is hoped that the direction of progress presented in this paper can offer an example of how schools mathematics projects can develop.

The reader can thus discern two 'tracks' in this paper. The track at the foreground shows the mathematics involved in the solution strategies to the problem; the other parallel track models the underlying process one can use when faced with a mathematics problem to solve. The original problem is stated below:

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Generalizations of a Geometric Problem

In $\triangle ABC$, the altitude AD, the angle bisector AE and the median AF divide $\angle BAC$ into four equal parts. Prove that $\angle BAC = 90°$. (Weeks [1]; 1961; Page 257). Now it is restated as follows.

Problem 1. Given ABC is a triangle, $AD \perp BC$, AE is the bisector of $\angle BAC$ and F is the midpoint of BC, $\angle BAF = \angle FAE = \angle EAD = \angle DAC$. Prove that $\angle BAC = 90°$.

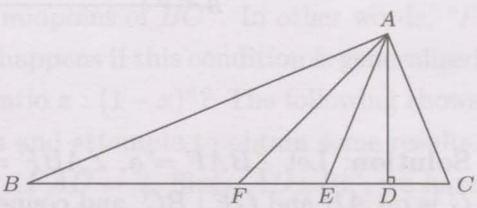

Figure 1.

This paper will not provide new solutions directly to Problem 1, since ten solutions have been given in [2]. But in the following sections, we shall generalize this problem, and provide solutions to new problems. The solutions to those new problems are also solutions to Problem 1.

2. Generalization

A close examination of one solution of Problem 1 in [2] reveals that not all the conditions were used in the proof. In particular, a revision of the original condition

$$\angle BAF = \angle FAE = \angle EAD = \angle DAC$$

to a *weaker condition* $\angle BAF = \angle DAC$ will yield the same result. The original problem is therefore revised as follows.

Problem 2. Given ABC is a triangle, $AD \perp BC$, F is the midpoint of BC and $\angle BAF = \angle DAC$. Prove $\angle BAC = 90°$.

Furthermore, we shall show that every condition in Problem 2 is necessary. So instead of proving Problem 2, we shall prove the following problem, by which the result of Problem 2 follows as a corollary.

Problem 3. Let ABC be a triangle, and D and F be two points on BC. Then any one of the four conditions below follows from the other three:

(1) $AD \perp BC$;

(2) F is the midpoint of BC;

(3) $\angle BAF = \angle DAC$; and

(4) $\angle BAC = 90°$.

Generalizations of a Geometric Problem

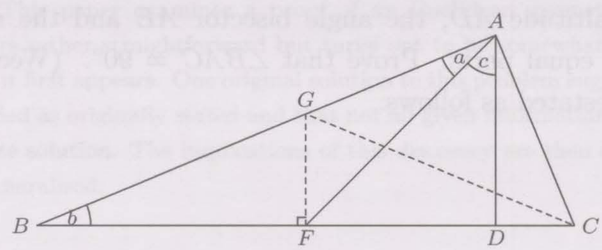

Figure 2.

Solution: Let $\angle BAF = a$, $\angle ABF = b$ and $\angle FAD = c$. Draw line GF such that G is on AB and $GF \perp BC$, and connect GC, as shown in Figure 2.

It suffices to show that

(i) under conditions (2) and (3), (1) and (4) are equivalent; and

(ii) under conditions (1) and (4), (2) and (3) are equivalent.

(i) Assume that conditions (2) and (3) are given. Since $\angle CAD = \angle BAF$, $\angle CAF = a + c$. Since GF is the perpendicular bisector of BC, $\angle CGF = \angle BGF$. Thus

$$\begin{aligned} AD \perp BC &\iff GF \parallel AD \ \text{(since } GF \perp BC\text{)} \\ &\iff \angle CGF = \angle BGF = \angle BAD = a + c = \angle CAF \\ &\iff A, G, F \text{ and } C \text{ are concyclic} \\ &\iff \angle GAC = \angle GFC = 90^\circ. \end{aligned}$$

Hence (1) and (4) are equivalent.

(ii) Assume that conditions (1) and (4) are given.

Since $\angle GFC = 90^\circ$ and $\angle BAC = 90^\circ$, the four points A, G, F and C are concyclic. Thus

$$\angle GCF = \angle BAF = a.$$

Since $AD \perp BC$ and $GF \perp BC$,

$$\angle ABC + \angle ACB = 90^\circ = \angle DAC + \angle ACB,$$

implying that

$$\angle DAC = \angle ABC = b.$$

Thus

$$\begin{aligned} BF = CF &\iff \triangle GFB \cong \triangle GFC \ \text{(since } GF \perp BC\text{)} \\ &\iff \angle GCF = \angle ABC \\ &\iff \angle DAC = b = \angle ABC = \angle GCF = a = \angle BAF. \end{aligned}$$

Hence (2) and (3) are equivalent. □

Generalizations of a Geometric Problem

3. Algebraic solutions

One way to extend a problem is to loosen some of the original limiting conditions and explore the results as a consequence of this 'adjustment'. This is the approach taken in the final section of this paper.

The reader will note that one of the necessary conditions for the same result to hold in both the above problems is that "F is the midpoint of BC". In other words, "F divides segment BC in the ratio 1 : 1". What happens if this condition is generalized to one where "F divides segment BC in the ratio $x : (1-x)$"? The following shows the formal statement of the extended problem and attempts to obtain some results.

Let $BC = 1$ and $BF = x$. So $0 < x < 1$. Let $\angle FAD = \beta$. Since $AD \perp BC$, we have $0 < \beta < 90°$. We try to express $\angle BAC$ as a function of x and β.

Problem 4. Given ABC is a triangle, $AD \perp BC$, and $\angle BAF = \angle DAC$. Let $x = \frac{BF}{BC}$ and $\angle FAD = \beta$. Prove that

$$\cot \angle BAC = (1-2x)\cot\beta.$$

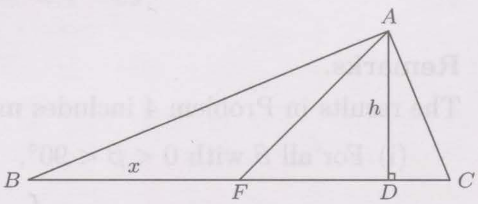

Figure 3.

Solution. Without loss of generality, we assume $BC = 1$ unit, and let $AD = h$ unit, let $\angle BAF = \angle CAD = \alpha$. So $BF = x$ unit. It can be seen that

$$BF = x = 1 - h\tan\alpha - h\tan\beta,$$

hence

$$\tan\alpha + \tan\beta = \frac{1-x}{h}. \tag{1}$$

From another perspective, x can also be seen as $BD - FD$, so that

$$\begin{aligned} x &= h\left(\tan(\alpha+\beta) - \tan\beta\right), \\ h &= \frac{x}{\tan(\alpha+\beta) - \tan\beta}. \end{aligned}$$

Substituting this expression of h into (1), we have

$$\begin{aligned} \tan\alpha + \tan\beta &= \frac{1-x}{x}\left(\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} - \tan\beta\right) \\ &= \frac{(1-x)\tan\alpha(1+\tan^2\beta)}{x(1-\tan\alpha\tan\beta)}. \end{aligned}$$

Generalizations of a Geometric Problem

Upon further simplifying,

$$\begin{aligned}
\tan\alpha(1-2x+\tan^2\beta) &= x\tan\beta(1-\tan^2\alpha)\\
\frac{\tan\alpha}{1-\tan^2\alpha} &= \frac{x\tan\beta}{1-2x+\tan^2\beta}\\
\tan 2\alpha &= \frac{2x\tan\beta}{1+\tan^2\beta-2x}.
\end{aligned}$$

Then using the addition formula:

$$\begin{aligned}
\tan(2\alpha+\beta) &= \frac{\tan 2\alpha+\tan\beta}{1-\tan\alpha\tan 2\beta}\\
&= \frac{\tan\beta}{1-2x}.
\end{aligned}$$

The details in the above proof are easy to verify and left to the readers. □

Remarks.

The results in Problem 4 includes more information than Problem 2.

(i) For all β with $0<\beta<90°$,

$$\angle BAC \begin{cases} =90°, & \text{if } x=1/2;\\ >90°, & \text{if } x>1/2;\\ <90°, & \text{if } x<1/2.\end{cases}$$

(ii) Since $0<x<1$, we have $-\cot\beta<\cot\angle BAC<\cot\beta$. Thus

$$\beta<\angle BAC<180°-\beta.$$

(iii) When x tends to 0, $\angle BAC$ approaches β; when x tends to 1, $\angle BAC$ approaches $180°-\beta$.

In the end of this section, we show that the two variables 'x' and 'β' in Problem 4 are independent, provided the conditions that $0<x<1$ and $0°<\beta<90°$.

Let x and β be any variables with $0<x<1$ and $0°<\beta<90°$. We can construct a triangle ABC with a point F on BC such that $BF:BC=x:1$ and $\angle FAD=\beta$, where AD is the height of $\triangle ABC$ on the side BC.

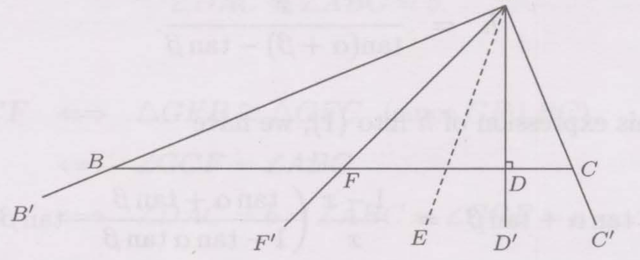

Figure 4.

Generalizations of a Geometric Problem

Step 1: Construct $\angle B'AC'$ such that $\cot \angle B'AC' = (1-2x)\cot\beta$. Since $0 < x < 1$, we have $\beta < \angle B'AC' < 180^\circ - \beta$.

Step 2: Draw the bisector AE of $\angle B'AC'$.

Step 3: Draw $\angle F'AD'$ within $\angle B'AC'$ such that $\angle F'AD' = \beta$ and AE is also the bisector of $\angle F'AD'$.

Step 4: Draw any line BC such that $BC \perp AD'$, cutting AB', AF', AD' and AC' at points B, F, D and C, respectively. Since $\angle B'AD' < 90^\circ$, such a line BC exists.

The above construction yields that $\cot \angle BAC = (1-2x)\cot\beta$, $\angle FAD = \beta$, $AD \perp BC$ and $\angle BAF = \angle CAD$. If $BF : BC = y$, it follows from Problem 4 that

$$1 - 2y = \frac{\cot \angle BAC}{\cot \beta} = 1 - 2x,$$

implying that $y = x$. Hence $BF : BC = x$.

References

[1] Arthur W. Weeks and Jackson B. Adkins, *A course in geometry, plane and solid*, Boston, Ginn [1961].

[2] R. Lancaster (Ed.), Ten solutions to a very special geometry problem, *Ontario Mathematics Gazette* 28(3) (1990), 33-39.

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