Classroom Corner

Why Area of $\triangle ABC = \frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_A \\ y_A & y_B & y_C & y_A \end{vmatrix}$?

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Since the day I was taught to find the area of a triangle, when given the coordinates of the three vertices, by the formula:

$$\text{Area of } \triangle ABC = \frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_A \\ y_A & y_B & y_C & y_A \end{vmatrix},$$

where the coordinates of A, B and C are (x_A, y_A), (x_B, y_B) and (x_C, y_C) respectively, I was very puzzled and rather upset. Why? Anyone would ask. The reason is simple. I do not know why this formula works! Even after teaching for almost nine years, I still do not know the answer. Recently, I asked myself why I still do not know the answer and the only excuse I could provide was this is a formula taught in secondary school and at 'A' levels, we assume that students already know this formula and hence I did not give it a deeper thought to unveil the reason behind.

However, that was just an excuse to make myself feel better for a while. I was still very dissatisfied with myself. So I thought through all the mathematics concepts I learnt starting from primary school and tried to build this formula from as basic as possible. Finally with GOD's grace, I discover the following:

From the definition of cross product in vectors we have $\mathbf{a} \wedge \mathbf{b} = (|\mathbf{a}| \cdot |\mathbf{b}| \sin\theta)\hat{\mathbf{n}}$, where $\theta = \angle BAC$. Hence,

$$\text{Area of } \triangle ABC = \frac{1}{2}|\mathbf{AB}| \cdot |\mathbf{AC}| \sin\theta = \frac{1}{2}|\mathbf{AB} \wedge \mathbf{AC}|.$$

The expression $\mathbf{AB} \wedge \mathbf{AC}$ can be simplified as $\mathbf{OA} \wedge \mathbf{OB} + \mathbf{OB} \wedge \mathbf{OC} + \mathbf{OC} \wedge \mathbf{OA}$, where $\mathbf{O}$ is the origin. So the area of $\triangle ABC$ can be written as

$$\frac{1}{2}|\mathbf{OA} \wedge \mathbf{OB} + \mathbf{OB} \wedge \mathbf{OC} + \mathbf{OC} \wedge \mathbf{OA}|.$$

Please note that in this formula A, B and C can be cyclically interchanged i.e. A changes to B, B changes to C and C changes to A. If one started with point A, then it would end up with point A. In general, if given the points $A(x_A, y_A, z_A)$,

Classroom Corner

$B(x_B, y_B, z_B)$ and $C(x_C, y_C, z_C)$, then the area of $\triangle ABC$ can be written as

$$\frac{1}{2}\left|\begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix} \wedge \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix} + \begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix} \wedge \begin{pmatrix} x_C \\ y_C \\ z_C \end{pmatrix} + \begin{pmatrix} x_C \\ y_C \\ z_C \end{pmatrix} \wedge \begin{pmatrix} x_A \\ y_A \\ z_A \end{pmatrix}\right|.$$

If points A, B and C are on the x-y plane then $z_A = z_B = z_C = 0$. So we can write the area of $\triangle ABC$ as

$$\frac{1}{2}\left|\begin{pmatrix} x_A \\ y_A \\ 0 \end{pmatrix} \wedge \begin{pmatrix} x_B \\ y_B \\ 0 \end{pmatrix} + \begin{pmatrix} x_B \\ y_B \\ 0 \end{pmatrix} \wedge \begin{pmatrix} x_C \\ y_C \\ 0 \end{pmatrix} + \begin{pmatrix} x_C \\ y_C \\ 0 \end{pmatrix} \wedge \begin{pmatrix} x_A \\ y_A \\ 0 \end{pmatrix}\right|,$$

and after some manipulations we get

$$\frac{1}{2}\left|\begin{pmatrix} 0 \\ 0 \\ x_A y_B - y_A x_B \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_B y_C - y_B x_C \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ x_C y_A - y_C x_A \end{pmatrix}\right|,$$

which is simplified to the notation that was taught to us:

$$\frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_A \\ y_A & y_B & y_C & y_A \\ 0 & 0 & 0 & 0 \end{vmatrix} \text{ i.e. } \frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_A \\ y_A & y_B & y_C & y_A \end{vmatrix}.$$

So the next obvious question we shall ask ourselves is what then should the notation for the area of $\triangle ABC$ if points A, B and C are general points in three dimensional?

The obvious extension for $\triangle ABC$ is

$$\frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_A \\ y_A & y_B & y_C & y_A \\ z_A & z_B & z_C & z_A \end{vmatrix}$$

and bearing in mind the evaluation of this notation is using the idea of cross-products. When further evaluated,

$$\frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_A \\ y_A & y_B & y_C & y_A \\ z_A & z_B & z_C & z_A \end{vmatrix} = \frac{1}{2}\left|\begin{pmatrix} y_A z_B + y_B z_C + y_C z_A - (z_A y_B + z_B y_C + z_C y_A) \\ -[x_A z_B + x_B z_C + x_C z_A - (z_A x_B + z_B x_C + z_C x_A)] \\ x_A y_B + x_B y_C + x_C y_A - (y_A x_B + y_B x_C + y_C x_A) \end{pmatrix}\right|.$$

Please note that the modulus on the right hand side of the above expression refers to the magnitude of the column vector.

After a few years of teaching, some students asked me why if points of $\triangle ABC$ are taken in anti-clockwise direction, the answer in the modulus is positive and when taken in clockwise direction, the answer in the modulus will be negative. My intuition was this should be consistent with the convention we have in trigonometry.

Classroom Corner

Let's consider figure 1. If we use clockwise direction, say point A to B to C and back to A, then the notation to represent the area of $\triangle ABC$ is

$$\frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_A \\ y_A & y_B & y_C & y_A \end{vmatrix},$$

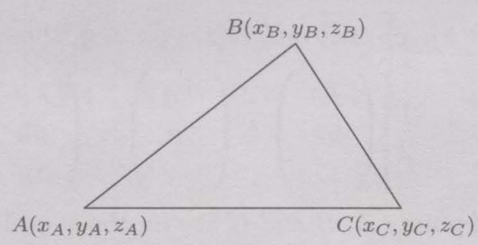

Figure 1.

which if we refer back is $\frac{1}{2}|\mathbf{AB} \wedge \mathbf{AC}|$. Recall that cross-product is also call vector product i.e. there is "direction" within the modulus sign. With respect to figure 1, by applying the right hand rule, the direction for $\mathbf{AB} \wedge \mathbf{AC}$ is into and perpendicular to the plane ABC i.e. in the opposite direction of $\mathbf{k}$. Notice that the angle involved is going clockwise and since the cross-product formula involves the sine function, my intuition was correct.

The next two questions one may ask will be how to extrapolate the formula

$$\frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_A \\ y_A & y_B & y_C & y_A \\ z_A & z_B & z_C & z_A \end{vmatrix}$$

to a quadrilateral and why the formula works.

Let's consider the quadrilateral in figure 2. If we consider clockwise direction point A to B to C to D and back to A, then mathematically is its area represented by

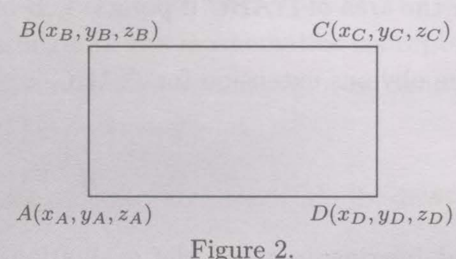

Figure 2.

$$\begin{aligned} & \frac{1}{2}|\mathbf{AB} \wedge \mathbf{AC}| + \frac{1}{2}|\mathbf{AC} \wedge \mathbf{AD}| \\ = \; & \frac{1}{2}|\mathbf{AB} \wedge \mathbf{AC} + \mathbf{AC} \wedge \mathbf{AD}| \quad \text{(why?)} \\ = \; & \vdots \\ = \; & \frac{1}{2}|\mathbf{OA} \wedge \mathbf{OB} + \mathbf{OB} \wedge \mathbf{OC} + \mathbf{OC} \wedge \mathbf{OD} + \wedge \mathbf{OD} \wedge \mathbf{OA}| \\ = \; & \frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_D & x_A \\ y_A & y_B & y_C & y_D & y_A \\ z_A & z_B & z_C & z_D & z_A \end{vmatrix} \quad \text{(why?)} \end{aligned}$$

Classroom Corner

In fact by this argument, one can prove by induction that the extension of this formula can be applied to a n-gon but bearing in mind that all the points MUST be considered in "clockwise" or "anticlockwise" direction. Clearly, in our discussion, the n-gon must lie on the same plane.

The next question I asked myself was:

If points are not considered in "clockwise" or "anticlockwise" direction then is there any geometrical meaning in applying the formula

$$\frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_D & x_A \\ y_A & y_B & y_C & y_D & y_A \\ z_A & z_B & z_C & z_D & z_A \end{vmatrix}?$$

Let's consider $A(0,0)$, $B(0,6)$, $C(6,4)$ and $D(6,4)$ as in figure 3. When I was taught the formula

$$\triangle ABC = \frac{1}{2}\begin{vmatrix} x_A & x_B & x_C & x_A \\ y_A & y_B & y_C & y_A \end{vmatrix},$$

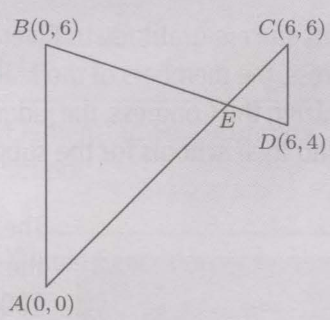

Figure 3.

I was very amazed and my intuition tells me if instead of considering point A to B to C to D and back to A, I consider point A to B to D to C and back to A then with respect to figure 3 the answer obtained will be the area of $\triangle ABE+$ area of $\triangle ECD$.

Let's proceed and verify. By simple calculation one can easily find that the x-coordinate of point E is $\frac{9}{2}$. Hence the area of $\triangle ABE+$ area of $\triangle ECD = \frac{1}{2}(6)(\frac{9}{2}) + \frac{1}{2}(\frac{3}{2})(2) = 15$ sq units.

However if instead by considering point A to B to D to C and back to A and modified the formula we get

$$\frac{1}{2}\begin{vmatrix} 0 & 0 & 6 & 6 & 0 \\ 0 & 6 & 4 & 6 & 0 \end{vmatrix} = 12 \text{ sq units.}$$

Bad luck, this round my intuition fails. So I am unable to offer any geometrical interpretation if vertices are not considered in "clockwise" or "anti-clockwise" direction.

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