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Will the Sun Rise Again?

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Will the Sun Rise Again?

If the Sun has risen on n successive days, what is the PROBABILITY that it will rise on the next day?

Laplace gave the answer: $\frac{n+1}{n+2}$. For $n = 1$, this is $\frac{2}{3}$. Since he assumes that it is "equally likely" that the Sun rises or not on each day, how come the answer is not $\frac{1}{2}$?

In the Urn model, there are two balls either black or white. Draw the first ball and see that it is white. What is the probability that the other ball (in the urn) is also white? The urn may contain i black balls, $i = 0, 1, 2$. Laplace calls these possibilities "causes", and I will denote them by C_i. ("Notation can be incredibly important!") Denote by E the "event" that the first ball drawn is white, and by F the event that the second ball be also white. Then we have the conditional probabilities:

$$P(E|C_i) = \frac{2-i}{2}, \qquad i = 0, 1, 2.$$

Laplace argues that the "inverse probabilities" should be proportional to the conditional probabilities above. This is known as Bayes's Rule. Put

$$S = \sum_i P(E|C_i) \tag{1}$$

then

$$P(C_i|E) = P(E|C_i)/S. \tag{2}$$

The reader can now compute the required

$$P(F|E) = \sum_i P(C_i|E)P(F|C_i \cap E) \tag{3}$$

and be satisfied with Laplace's announcement. [1]

In 1935 I was in my last ("senior") year of high school at Li Da Academy (立達學園) in Shanghai, China. Liu Bing Zhen (劉炳震) (Liu is the family name) was a junior. We learned Laplace's Sunrise Theorem in an Algebra class. "Probability", following "Permutations and Combinations" were part of the Algebra curriculum. See for instance the marvellous text by Fine (after whom the old venerable Fine Hall in Princeton University was named). This book was adopted as textbook at many high schools in China then, after certain more difficult British books.

The demonstration of the result by the teacher did not satisfy us. So we decide to work it out ourselves. After some struggling we got the results, in fact in a general form that we found in a book by 王星拱 on scientific methods. (Recently I tried to obtain a copy of this book from Chinese libraries but so far without success.) Here is an intermediate result that is the heart of the matter.

An urn contains black and white balls in unknown numbers. Draw n balls in succession (without putting any back) and see that $n-r$ of them are white, $0 \le r \le n$. What is the probability that the next ball drawn, supposing of course the urn is not empty, be white? The answer is $\frac{n-r+1}{n+2}$. For $r=0$, this is the previous case.

Let the total number of balls in the urn be $n+m$, $m \ge 1$. Denote by C_i the Laplacian *cause* that the total number of black balls in the urn is $r+i$. Denote by E the event that the first n balls drawn contain exactly r black balls; and by F the event that the $(n+1)$ st ball drawn be white. We can compute the various conditional probabilities as in the previous simplest case. Now we need to learn to count permutations and combinations, see [1] or any elementary text such as [3]. The results are recorded below for the reader to verify. The symbol S is defined as in (1) with the new range

[1] In deriving equation (2), all $P(C_i)$ are equal. This follows from the assumption of "total ignorance of the possible causes."

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$0 \leq i \leq m.$

$$P(E|C_i) = \binom{n}{r}\binom{m}{i}\binom{n+m}{r+i}^{-1};$$

$$P(F|E \cap C_i) = \binom{m-1}{i}\binom{m}{i}^{-1}.$$

After obvious manipulations, we get

$$S = \binom{n+m}{n}^{-1} \sum_{i=0}^{m} \binom{r+i}{r}\binom{n+m-r-i}{n-r}. \tag{4}$$

The sum above is evaluated by the following

Main Identity. For nonnegative integers x, y, z and i:

$$\sum_{i=0}^{z} \binom{x+i}{x}\binom{y+z-i}{y} = \binom{x+y+z+1}{x+y+1}. \tag{5}$$

Using (5) in (4) we obtain

$$\binom{n+m}{n}^{-1}\binom{n+m+1}{n+1} = \frac{n+m+1}{n+1}. \tag{6}$$

Now we put

$$f(n,m) = \binom{n}{r}^{-1} S.$$

Then "it is easy to see" (as Laplace would certainly say) that

$$P(F|E) = \frac{f(n+1, m-1)}{f(n,m)}$$

which after "incredible" cancellations, reduces to the desired result: $\frac{n-r+1}{n+2}$

Isn't it a Miracle that the arbitrarily introduced number m disappears in the final result? Can we prove this *a priori*?

It remains to prove (5). We begin with the minimalist identity

$$\binom{x+i}{x} = \binom{x-1+i}{x-1} + \binom{x-1+i}{x}. \tag{7}$$

This is given in Fine's fine book, p. 404, with a neat counting argument (let's not spoil it by a dumb computation!). By repetition (induction) we have

$$\binom{x+i}{x} = \sum_{j=0}^{i} \binom{x-1+j}{x-1}. \tag{8}$$

Now we substitute (8) in (5) to convert the simple summation into a double or iterated summation, then reverse the order of the summation and use (8) again to reduce the reversed iterated summation to a simple one again. Thus the sum on the left side of (5) is equal to

$$\sum_{i=0}^{z}\left\{\sum_{j=0}^{i}\binom{x-1+j}{x-1}\right\}\binom{y+z-i}{y} = \sum_{j=0}^{z}\sum_{i=j}^{z}\binom{x-1+j}{x-1}\binom{y+z-i}{y}$$
$$= \sum_{j=0}^{z}\binom{x-1+j}{x-1}\binom{y+1+z-j}{y+1}.$$

The last-written sum is nothing but the (first) sum in (5) with (x, y) replaced by $(x-1, y+1)$. Continuing this process, the original (x, y) can be reduced to $(0, y+x)$. The corresponding sum is then

$$\sum_{i=0}^{z}\binom{0+i}{0}\binom{y+x+z-i}{y+x} = \sum_{i=0}^{z}\binom{y+x+z-i}{y+x}$$

which is equal to the right side of (5), by (8). Eureka! We were quite pleased with our method and named it céng tùi fă (層蜕法).

We told our results to Mr Mao (毛路真) an instructor at Chekiang University in Hangchow (杭州) where my home was (now Hangzhou). He found in Todhunter's History [3; p.454ff] that Prevost and Lhuilier had proved Laplace's Sunrise Theorem with the urn model. Laplace did not do so, but took an easy way out by assuming a continuum of a priori probabilities as a density p, $0 < p < 1$, and computed an integral. An exposition of this is given in [2; p.123]; a variation of this is given in [4; p.123].

In 1936 both Liu and I entered Tsinghua University in Peking (Beijing). His admission was a special case since he had not graduated from high school. We did not see each other often during the school year. Then one day toward the end of the semester, the news spread that a body had been found on the railway tracks behind the campus, and was identified as Liu's. I went with some others to the spot but refrained from looking for the remains. That evening, Old Wang our dormitory steward knocked at my door and handed me a letter. It was from Liu and began with: "I have decided to commit suicide". He left me some books including Bôcher's Higher Algebra (in Chinese). Two letters were found on his body, one from an aunt who scolded him for spending too much money on his meager inheritance, the other from a girl schoolmate at Li Da whom he must have been "dating".

I wrote up our result and submitted it to the newly established Chinese Journal of Mathematics. It was published in Volume 1, Number 4, the penultimate issue of that journal. In 1937 Japan began its invasion of China and the journal died. Our article has recently been translated by my *ancien élève* Elton Pei Hsu (徐佩) under the title "Elementary proof of a theorem in probability" (to appear).

Mr Liu and I were actually classmates in Hangchow High School as "freshmen". His father Liu Da Bai (劉大白) was a renowned poet and served as (deputy?) minister of education. He was so "frugal and clean" (廉潔) that he left little inheritance. My father knew and admired him. Bing Zhen visited us often at No. 17, Wushan Road, Hangchow by suddenly appearing at the back door. He had a broad, bright and pale face, and a wobbly gait; was nicknamed "beat devil" (打鬼) by his classmates. Once he caused a stir in our geography class when he dared to question the teacher's accuracy in naming the thirteen American states which initiated the American Revolution to form The United States. He was dismissed after one year on account of poor grades and ill behavior. I had been reprimanded more than once by the "guardian" when he caught me wearing a long gown instead of the semi-military yellow uniform as per new regulation. It was rumored that I was saved from dismissal by some teachers including Mr Chow Ming Sen (周明生) who taught Chinese. I stayed on for another half-year, then left voluntarily. Before I went to Li Da (which had a reputation of being "liberal" then), I made a train trip (about four hours) to see Liu in Shanghai. It is *très probable* that we went to a special section of a big department store to look at foreign books.

A few years ago Prof Bernard Bru was kind enough to send me a copy of the article by Prevost and Lhuilier: "Sur les probabilités" in the Memoires de l'Acadámie (Paris), 1799, pp.117-142; read in 1795, shortly after the French Revolution. It was written in long hand, without the symbols for factorial or binomial coefficients. After deciphering, it began with a main Lemma that turned out to be the identity (5). Four particular cases were expounded, followed by these words (in my faithful translation): "I omit the general development which presents no difficulty, being entirely similar to the preceding examples." Of course they did have the result, but apparently it was Bishop Terrot in 1853 who proved the identity by use of the generating function

$$(1-t)^{-x-1} = \sum_{i=0}^{\infty} \binom{x+i}{x} t^i \tag{10}$$

(in some form). Then (5) follows by checking the coefficient of t^z in

$$(1-t)^{-x-1}(1-t)^{-y-1} = (1-t)^{-(x+y+1)-1}.$$

Attention: those "minus one"s! While drafting the present article. I made the discovery that the identity (5) is hidden on p.190 of [2] as an exercise, scarcely recognizable in disguise. "Notation can be incredibly frustrating!"

The story has another happy ending: (5) can be proved by smart COUNTING, as noted above for (8) which is in fact a special case.

Choose, in increasing order, $x+y+1$ integers from the integers from 1 to $x+y+z+1$. Let the $(x+1)$ st choice be $x+i+1$, then $0 \le i \le z$. For each i, x integers are chosen from 1 to $x+i$. The other y integers are chosen from $x+i+2$ to $x+y+z+1$. The total number of these choices is the i th term of the sum in (5). Q.E.D. See [5]. I owe Elton for this reference.

References

[1] Henry Burchard Fine: A College Algebra, Ginn & Co., 1901.

[2] Kai Lai Chung: Elementary Probability Theory with Stochastic Processes, Third Edition, Springer Verlag, 1979.

[3] I. Todhunter: A History of the Mathematical Theory of Probability, MacMillan Co. 1865.

[4] William Feller: An Introduction to Probability Theory and Its Applications, Third Edition, Wiley & Sons, 1968.

[5] L. Lovasz: Problems and Exercises, North Holland Publ. Co., 1979; See Problem 42(i).