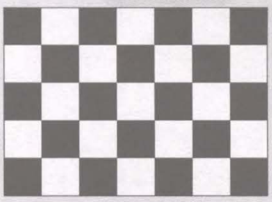
A. Prized Problems

Problem 1

Given a (2m+1) x (2n+1) checkerboard with black squares at the four corners (for example, a 5 x 7 checkerboard is shown in the diagram below), show that if any one white square and two black squares are removed, the remaining board can be covered with 1 x 2 and 2 x 1 rectangles (i.e. ▭▭ and ⊟).



(Book voucher up to $150)

Problem 2

Let a, b and c be positive integers. Suppose today is a Sunday. What day is it after ${}_{a}b^{c}$ days?

(Book voucher up to $150)

B. Instruction

1. Prizes in the form of book vouchers will be awarded to one or more received best solutions submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

3. Solutions should be sent to:
The Editor, Mathematical Medley,
c/o Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore117543 ; and should arrive before 1 May 2003.
Alternatively, softcopies of the solutions can also be sent to the email address: mattanv@nus.edu.sg.

4. The Editor's decision will be final and no correspondence will be entertained.

SOLUTIONS

C. Solutions to the problems of volume 29, No.1, 2002

Problem 1.

Show that every positive integer can be written as

$$a_1 1^3 + a_2 2^3 + \cdots + a_k k^3$$

for some positive integer k and $a_i \in \{\pm 1, \pm 2\}$.

(One $150 book voucher)

Solution

by Leung Ngai-hang Zachary – Anglo Chinese Junior College

Let A be the set of all numbers which can be expressed in the form

$$a_1 1^3 + a_2 2^3 + \ldots a_k k^3$$

for $a_i \in \{\pm 1, \pm 2\}$.

For simplicity, let us denote the sum $a_1 1^3 + a_2 2^3 + \ldots a_k k^3$ by $(a_1, a_2, a_3, \ldots, a_k)$.

We can easily verify the following congruences modulo 18:

$0 \equiv (2,2)$	$1 \equiv (1)$
$2 \equiv (2)$	$3 \equiv (2,-1,1)$
$4 \equiv (-2,1,-1,1)$	$5 \equiv (-1,1,-1,1)$
$6 \equiv (-2,1)$	$7 \equiv (-1,1)$
$8 \equiv (1,2,-1)$	$9 \equiv (1,1)$

(-1 to -8 can be obtained by reversing the sign of each number.)

We shall now show that if n is in A, then $n \pm 18$ is also in A.

Suppose n is in A. Then $n = (a_1, a_2, \ldots, a_k)$ for suitable $a_i \in \{\pm 1, \pm 2\}$.

Using the identity $(k-1)^3 - 2k^3 + (k+1)^3 = 6k + 2$, we see that

$(a_1, a_2, \ldots, a_k, -1, 2, -1, 1, -2, 1)$

$= (a_1, a_2, \ldots, a_k) - 6(k+2) - 2 + 6(k+5) + 2$

$= n + 18$

Similarly, $(a_1, a_2, \ldots, a_k, 1, -2, 1, -1, 2, -1) = n - 18$

Therefore, if n is in A, $n + 18k$ is also in A for any integer k.

For any number x, there is a number n in A congruent to x modulo 18.

Thus $x = n + 18k$ for some integer k is in A.

Editor's note: Zachary gave an elegant proof for the problem. Since he is the only one sent in the solution to this problem, the prize went to him.

SOLUTIONS

C. Solutions to the problems of volume 29, No.1, 2002

Problem 2.

The n^{th} subdivision of an equilateral triangle is the configuration obtained by

(i) dividing each side of the triangle into n equal parts by (n-1) points and

(ii) adding 3(n-1) line segments to join the 3(n-1) pairs of points in (i) on adjacent sides so that the line segments are parallel to the third side. For example,

n = 2

n = 3

How many triangles can you find in the 10th subdivision of an equilateral triangle?

(One $150 book voucher)

Solution

by Tan Weiyu, Colin – Raffles Junior College

Any triangle is determined by three pairwise intersecting lines, i.e. the three lines formed by extending the three sides of the triangle. In the 10^{th} subdivision of an equilateral triangle (call the triangle T) all line segments are parallel to one of the sides of T. Thus all triangles that can be found are either equilateral triangles pointing up (Δ) or down (∇). We shall count them separately.

Let T have side length 10. Call an equilateral triangle of side length k as having size k. It can be verified that there are $1+2+\cdots+(11-k)=t_{11-k}$ (where t_k denotes the kth triangular number) triangles of size k pointing up, for $k=1,\ldots,10$ (see note on the next page). Thus there are in total $\sum_{k=1}^{10} t_{11-k}=\sum_{k=1}^{10} t_k=\frac{1}{2}\sum_{k=1}^{10}\left(k^2+k\right)=220$ triangles pointing up.

Similarly, it can be verified that there are $1+2+\cdots+(11-2k)=t_{11-2k}$ triangles of size k pointing down, $k=1,\ldots,\left\lfloor\frac{10}{2}\right\rfloor=5$. Thus there are $\sum_{k=1}^{5} t_{11-2k}=\sum_{k=1}^{5} t_{2k-1}=\sum_{k=1}^{5}\left(2k^2-k\right)=95$ triangles pointing down.

Therefore, there are $220+95=315$ triangles in the 10^{th} subdivision of an equilateral triangle.

SOLUTIONS SOLUTIONS SOLUTIONS SOLUTIONS SOLUTIONS SOLUTIONS SOLUTIONS

C. Solutions to the problems of volume 29, No.1, 2002

Note: It is a matter of counting to verify that there are t_{11-k} (resp. t_{11-2k}) triangles of size k pointing up (resp. down). To do this, we introduce some notation: Let the line segments with one point on each side of T excluding the base of T be labelled $l_1,\ldots,l_{10}$ respectively, starting from the top. Let l_0 denote the top vertex of T.

For the triangles pointing up, if the top vertex is on l_s, then the base is on l_{s+k}. Since $s \geq 0$ and $s+k \leq 10$, s ranges from 0 to $10-k$. On l_s, the top vertex can take $s+1$ positions and each position corresponding to one triangle. Thus there are $\sum_{s=0}^{10-k} s+1 = t_{11-k}$ triangles.

For the triangles pointing down, if the bottom vertex is on l_{s+k}, then the base is on l_s. Since $|l_s| = s \geq k$ (length of l_s must be longer than base) and $s+k \leq 10$, s ranges from k to $10-k$. On l_s, the base can take $s-k+1$ positions and each position corresponds to one triangle. Thus there are $\sum_{s=k}^{10-k} s-k+1 = t_{11-2k}$ triangles.

Editor's note: Solved also by Charmaine Sia Jia Min (Raffles Girls' School) and Joel Tan Wei En (Anglo Chinese School Independent). The prize is shared among Colin, Charmine and Joel.