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On Some Maximum Area Problems I

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1. Introduction

When the lengths of the three sides of a triangle are given as l_1, l_2 and l_3, then its area $\boldsymbol{A}$ is uniquely determined, and $\boldsymbol{A}=\sqrt{s(s-l_1)(s-l_2)(s-l_3)}$, where s is the semi-perimeter $\frac{1}{2}(l_1+l_2+l_3)$. This formula is usually called the Heron's formula. It is also well known that there is a unique circle circumscribing any given triangle.

However, when the number of sides of a plane polygon is more than three, the situation becomes much more complicated. For instance, consider all quadrilaterals of which the lengths of the four sides are given as l_1, l_2, l_3, and l_4. Since there are infinitely many such quadrilaterals, rather than considering the area of an individual quadrilateral, it is more natural to address the following questions: (1) Which of these quadrilaterals is cyclic? What is the radius of the circumscribing circle? (2) Which of these quadrilaterals achieves the maximum area? What is the formula of the maximum area? Is such a quadrilateral unique? In general, for any positive integer $n \geq 5$, similar questions can also be posed for n-gons with n side lengths prescribed.

In this paper, we shall present to readers the main results pertaining to these questions in a more systematic way, and try to provide the solutions to the problems in an easily accessible way. We shall first answer the questions mentioned above for quadrilaterals, then prove some results for general n-gons. To understand the discussion, the readers only need to have basic knowledge of calculus.

2. Conditions For A Quadrilateral To Achieve Maximum Area

Since we are mainly interested in those polygons which achieve maximum area and such polygons are clearly convex, thus in the following we shall only consider convex polygons.

We shall first consider the quadrilateral which achieves the maximum area. Let $ABCD$ be any quadrilateral whose four sides have lengths l_1, l_2, l_3 and l_4 (see Figure 1). Join AC and let ϕ and ψ denote the angles $\angle ABC$ and $\angle ADC$, respectively.

Thus, the area $\boldsymbol{A}$ of the quadrilateral $ABCD$ is given by:

$$\boldsymbol{A} = \tfrac{1}{2} l_1 l_2 \sin\phi + \tfrac{1}{2} l_3 l_4 \sin\psi . \tag{1}$$

In view of

$$\begin{aligned} AC^2 &= l_1^{\,2} + l_2^{\,2} - 2 l_1 l_2 \cos\phi \\ &= l_3^{\,2} + l_4^{\,2} - 2 l_3 l_4 \cos\psi, \end{aligned} \qquad \text{or} \qquad \cos\phi = k_1 + k_2 \cos\psi , \tag{2}$$

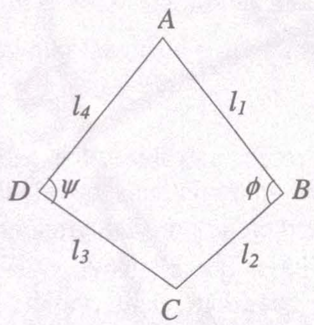

Figure 1

where

$$k_1 = \frac{l_1^{\,2} + l_2^{\,2} - l_3^{\,2} - l_4^{\,2}}{2 l_1 l_2}, \qquad k_2 = \frac{l_3 l_4}{l_1 l_2},$$

$$\sin\phi = \sqrt{1 - (k_1 + k_2 \cos\psi)^2} . \tag{3}$$

Substituting (3) into (1), we obtain

$$\boldsymbol{A} = \tfrac{1}{2} l_1 l_2 \sqrt{1 - (k_1 + k_2 \cos\psi)^2} + \tfrac{1}{2} l_3 l_4 \sin\psi .$$

Now $\boldsymbol{A}$ is a continuous function of the variable ψ and it is differentiable over $(0, \pi)$. In order to find out the value of ψ for which $\boldsymbol{A}$ attains the maximum value, we first find its stationary point(s) in $(0, \pi)$.

Now

$$\frac{d\mathbf{A}}{d\psi} = \frac{l_3 l_4}{2} \left[\cos\psi + \frac{(k_1 + k_2 \cos\psi)\sin\psi}{\sqrt{1 - (k_1 + k_2 \cos\psi)^2}} \right]. \tag{4}$$

Using (3) and (4), $l_3 \neq 0$, $l_4 \neq 0$, and $\dfrac{d\mathbf{A}}{d\psi} = 0$ we obtain

$$\sin\phi \cos\psi + (k_1 + k_2 \cos\psi)\sin\psi = 0 . \tag{5}$$

In view of (2), (5) becomes $\sin\phi\cos\psi + \cos\phi\sin\psi = 0$, that is,

$$\sin(\phi + \psi) = 0 . \tag{6}$$

Since

$$0 < \phi < \pi \quad \text{and} \quad 0 < \psi < \pi ,$$

equation (6) implies

$$\phi + \psi = \pi . \tag{7}$$

From (2) it follows easily that there is a unique ψ^* satisfying (7), that is $\boldsymbol{A}$ has a unique stationary point in $(0, \pi)$. Furthermore, for this ψ^*

$$\left.\frac{d^2 \mathbf{A}}{d\psi^2}\right|_{\psi=\psi^*} = -\frac{l_3 l_4}{2\sin\phi}\left(1 + \frac{l_3 l_4 \sin\psi}{l_1 l_2 \sin\phi}\right) < 0 \tag{8}$$

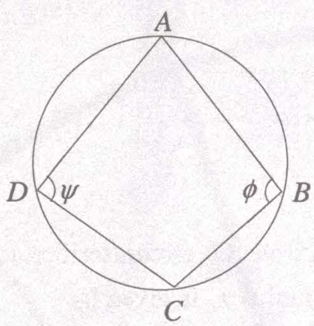

Figure 2

Thus from elementary calculus, it follows that

$\boldsymbol{A}$ attains the absolute maximum value at ψ^*.

Notice that equation (7) is actually equivalent to the condition that the quadrilateral $ABCD$ is cyclic, i.e., all its vertices lie on a circle as shown in Figure 2.

In the following, we shall show that when ϕ and ψ satisfy the equation $\phi + \psi = \pi$, the area of the quadrilateral $ABCD$ is given by

$$\boldsymbol{A} = \sqrt{(s - l_1)(s - l_2)(s - l_3)(s - l_4)} \tag{9}$$

where

$$2s = \sum_{i=1}^{i=4} l_i . \tag{10}$$

To derive (9), we employ (7) in (1) and (2), yielding

$$\boldsymbol{A} = \tfrac{1}{2}(l_1 l_2 + l_3 l_4)\sin\phi , \tag{11}$$

and

$$\cos\phi = \frac{l_1^{\,2} + l_2^{\,2} - l_3^{\,2} - l_4^{\,2}}{2(l_1 l_2 + l_3 l_4)} . \tag{12}$$

Hence $\sin\phi = \sqrt{1-\cos^2\phi} = \dfrac{\sqrt{\{(l_1+l_2)^2-(l_3-l_4)^2\}\cdot\{(l_3+l_4)^2-(l_1-l_2)^2\}}}{2(l_1l_2+l_3l_4)}$, (13)

and (9) follows immediately from (11) after the replacement of $\sin\phi$ by its expression in (13).

The above results are summarized in the following proposition:

Proposition 1. A quadrilateral with designated side lengths l_1, l_2, l_3 and l_4 achieves the maximum area if and only if it is a cyclic quadrilateral, and in this case the maximum area $\boldsymbol{A}$ is given by

$$\boldsymbol{A} = \sqrt{(s-l_1)(s-l_2)(s-l_3)(s-l_4)},$$

where $s = \frac{1}{2}\sum_{i=1}^{i=4} l_i$.

Remark 1 It is well known (see [1], for example) that if a circle C_3 circumscribes a triangle ABC whose three sides are of lengths l_1, l_2 and l_3, then its radius r_3 is given by

$$r_3 = \frac{l_1l_2l_3}{4\sqrt{s(s-l_1)(s-l_2)(s-l_3)}}.$$

Now suppose a circle C_4 circumscribes a quadrilateral whose four sides are l_1, l_2, l_3 and l_4, then we show that the radius r_4 of C_4 is given by

$$r_4 = \frac{\sqrt{(l_1l_2+l_3l_4)(l_1l_3+l_2l_4)(l_1l_4+l_2l_3)}}{4\sqrt{(s-l_1)(s-l_2)(s-l_3)(s-l_4)}}.$$

From (12), $\cos\phi = \dfrac{l_1^2+l_2^2-l_3^2-l_4^2}{2(l_1l_2+l_3l_4)}$,

$\cos\phi_1 = \dfrac{l_1}{2r}$, $\cos\phi_2 = \dfrac{l_2}{2r}$ (see Figure 3).

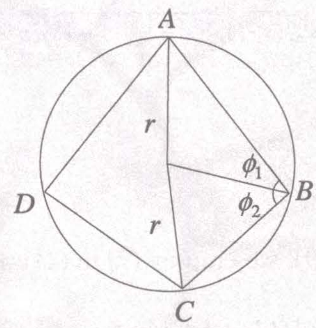

Figure 3

But $\cos\phi = \cos(\phi_1+\phi_2) = \cos\phi_1\cos\phi_2 - \sin\phi_{1_1}\sin\phi_2$, so

$$\frac{l_1^2+l_2^2-l_3^2-l_4^2}{2(l_1l_2+l_3l_4)} = \frac{l_1l_2}{4r^2} - \frac{\sqrt{(4r^2-l_1^2)(4r^2-l_2^2)}}{4r^2}.$$

It follows that

$$\sqrt{(4r^2-l_1^{\,2})(4r^2-l_2^{\,2})} = \frac{l_1l_2(l_1l_2+l_3l_4)-2r^2(l_1^{\,2}+l_2^{\,2}-l_3^{\,2}-l_4^{\,2})}{(l_1l_2+l_3l_4)}. \tag{14}$$

On squaring both sides of (14) and rearranging terms, we obtain

$$r^2[(l_1^{\,2}+l_2^{\,2}-l_3^{\,2}-l_4^{\,2})^2-4(l_1l_2+l_3l_4)^2]$$
$$= l_1l_2(l_1l_2+l_3l_4)(l_1^{\,2}+l_2^{\,2}-l_3^{\,2}-l_4^{\,2})-(l_1^{\,2}+l_2^{\,2})(l_1l_2+l_3l_4)^2,$$

and hence

$$r_4 = r = \frac{\sqrt{(l_1l_2+l_3l_4)(l_1l_3+l_2l_4)(l_1l_4+l_2l_3)}}{4\sqrt{(s-l_1)(s-l_2)(s-l_3)(s-l_4)}}.$$

3. Conditions For An n-Gon To Achieve The Maximum Area

In this section, we consider the set of n-gons whose sides have fixed lengths $l_1, l_2, \ldots, l_n$. Using the results obtained in Section 2, we shall prove that:(1) An n-gon achieves maximum area if and only if its vertices lie on a circle; (2) The value of the maximum area is independent of the order in which the n sides of the polygon are arranged.

Proposition 2. An n-gon, with given side lengths $l_1, l_2, \ldots, l_n$, achieves maximum area if and only if it is cyclic.

Proof For the necessary part, consider a polygon $A_1A_2 \cdots\cdots A_n$ (see Figure 4) whose sides $A_1A_2, A_2A_3, \cdots, A_nA_1$ have lengths $l_1, l_2, \ldots, l_n$ respectively, and suppose that it achieves the maximum area. Assume that in this shape, A_1A_4 has length l.

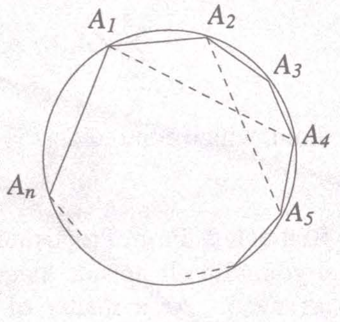

Figure 4

Consider the quadrilateral $A_1A_2A_3A_4$. If the n-gon $A_1A_2 \cdots\cdots A_n$ has achieved maximum area, then so has the quadrilateral $A_1A_2A_3A_4$ with $A_1A_4 = l$. From Section 2, it follows that vertices A_1, A_2, A_3 and A_4 must lie on the circumference of a circle, say C. Similarly, the vertices A_2, A_3, A_4, and A_5 of the quadrilateral $A_2A_3A_4A_5$ must also lie on the circumference of a circle, say D. However, as every three points determine a circle, C and D must be the same since they both contain the points A_2, A_3, and A_4. Inductively, it follows that every vertex A_i of the n-gon must lie on C.

To prove the sufficiency, let Q be an n-gon with side lengths $l_1, l_2, ..., l_n$ whose vertices $A_1, A_2, ..., A_n$ lie on a circle C with centre O and radius r. Intuitively, there exists an n-gon that achieves the maximum area. Let $\boldsymbol{P}$ be such a polygon whose n vertices are $B_1, B_2, ..., B_n$. Then, by the necessity part, $\boldsymbol{P}$ is circumscribed by a circle C'. Suppose the radius of C' is r_n and the centre is O' (see Figure 5). If we can show that $r_n = r$, then the n-gon Q is congruent to the n-gon P, thus it also achieves the maximum area.

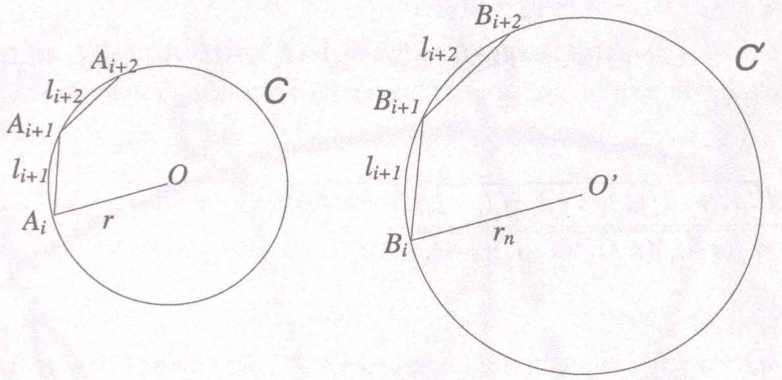

Figure 5

Suppose $r_n \neq r$, with no loss of generality, we assume that $r_n > r$. Then it follows that for each i $(i = 1, 2, \cdots, n)$, $\angle A_i A_{i+1} A_{i+2} < \angle B_i B_{i+1} B_{i+2}$, using the convention that $n+1=1$, $n+2=2$.

Thus

$$\sum_{i=1}^{i=n} \angle A_i A_{i+1} A_{i+2} < \sum_{i=1}^{i=n} \angle B_i B_{i+1} B_{i+2} . \tag{15}$$

However, both $\boldsymbol{P}$ and $\boldsymbol{Q}$ are n-gons, so

$$\sum_{i=1}^{i=n} \angle A_i A_{i+1} A_{i+2} = (n-2)\pi \text{ and } \sum_{i=1}^{i=n} \angle B_i B_{i+1} B_{i+2} = (n-2)\pi$$

hold, which contradicts the inequality (15).

Remark 2 From Proposition 2, one can also observe that the maximum area achieved by an n-gon with all its side lengths specified is independent of the order in which its n sides are arranged. As a matter of fact, from Proposition 2, if an n-gon with given side lengths achieves the maximum area, then it must be cyclic and hence can be thought of as the sum of n isosceles triangles with the centre of the circle as one common vertex. It is then obvious that the area of the n-gon remains unchanged regardless of how the n sides are arranged.

4. Summary And Remarks

In this paper, we derived the formula of the maximum area achieved by a quadrilateral whose four sides are prescribed by using elementary calculus techniques. We also proved that an n-gon achieves the maximum area if and only if it is cyclic. The maximum area of a quadrilateral with its four sides prescribed can also be obtained using Brahmagupta's formula, which states that the area of a quadrilateral equals

$$\sqrt{(s-a)(s-b)(s-c)(s-d)-abcd\cos^2\left(\tfrac{A+B}{2}\right)}$$

where $a, b, c,$ and d are the side lengths of the quadrilateral, $s=\frac{1}{2}(a+b+c+d)$, and A, B are the angles between sides a and d, and sides b and c, respectively. The interested readers may like to derive this formula from first principle. For $n \geq 5$, we do not know of any closed form formula for the maximum area achievable by an n-gon with prescribed side lengths. Interested readers my like to read more about areas of cyclic polygons in Robbins (1995).

REFERENCES

[1] Porter, R.I., 1970, *Further Mathematics*, London: G. Bell & Sons, Ltd.

[2] Robbins, D.P., 1995, Areas of Polygons Inscribed in a Circle, *American Mathematical Monthly*, 102, pp523-530.