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Editor's note:
Starting from this issue of the Medley, we will publish a series of notes on Graphs and Their Applications by Prof Koh Khee Meng. In these notes, Prof Koh will introduce some basic concepts and fundamental results on Graph theory, and show also their applications to problems in other areas. In addition, relevant problems with different degree of difficulty will be set for interested readers to have a go at them.

Graphs and Their Applications (1)

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1. The Konigsberg Bridge Problem

In an odd city of Eastern Prussia, called Konigsberg, there was a river, called River Pregel, flowing through its centre. In the 18th century, there were seven bridges over the river connecting the two islands and two opposite banks separated by the river as shown in Figure 1.1. It was said that the people in the city had always amused themselves with the following problem: *Starting with any one of the places A, B, C or D as shown in Figure 1.1, is it possible to find a route which passes through each of the seven bridges once and exactly once, and return to where you start?*

Leonhard Euler (1707-1783) first studied for the clergy at the Swiss university in Besel. In there Johann Bernoulli noticed his talent in mathematics and encouraged him to change his career. Euler became one of the greatest mathematicians of all time and one of the founders of the field known as topology.

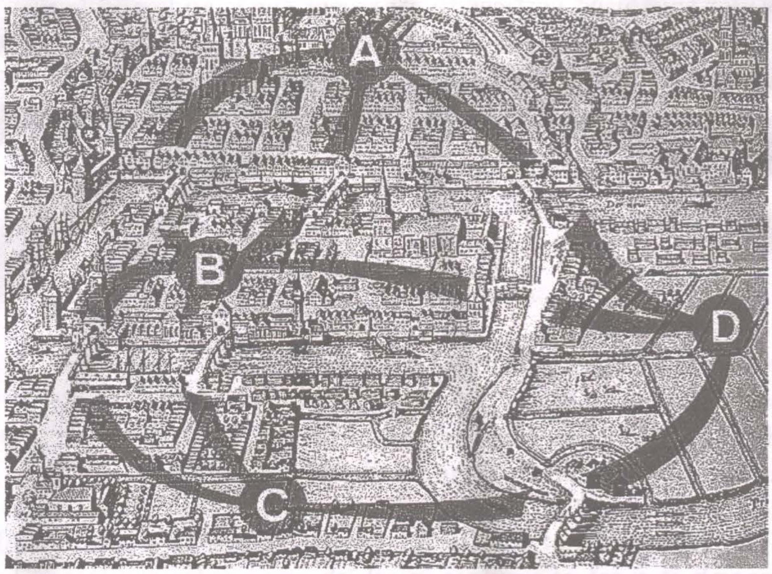


Figure 1.1 picture taken from Mathematics, Life Science Library, Time Inc.

No one could find such a route; and after a number of tries, many people believed that it is not possible, but no one could prove it either.

Leonhard Euler (1707-1783), the greatest mathematician that Switzerland has ever produced, was told of the problem. He found the problem interesting and realized that the problem is very much different in nature from problems in traditional geometry. He studied the problem and generalized it to a more general problem regardless of the number of islands and banks, and the number of bridges linking them, and finally solved the general problem. His finding was contained in an article entitled *"The solution of a problem relating to the geometry of position"* published in 1736. As a special case of his finding, he deduced the impossibility of finding such a route first time from mathematical point of view.

How did Euler generalize the Konigsberg Bridge Problem? How did he solve his more general problem? What is his finding? We shall answer these questions in due course.

2. Multigraphs and Graphs

Euler noticed that the Konigsberg Bridge Problem has nothing to do with traditional geometry where measurements of lengths and angles, and relative positions of vertices count. How big the islands and banks are, how long the bridges are, and whether an island is at the south or north of a bank are immaterial. The key factors are whether the islands or banks are linked by a bridge, and by how many bridges. Euler thus used vertices to represent the islands and banks, one for each island and bank, and two vertices are joined by k edges, where $k \geq 0$, when and only when the islands or banks represented the vertices are linked by k bridges. In particular, the situation for the Konigsberg Bridge Problem is represented by the model of Figure 2.1.

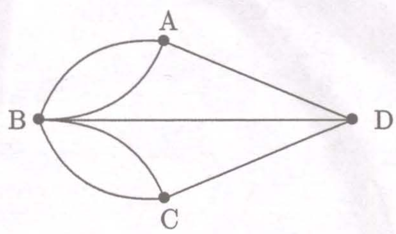


Figure 2.1

The model of Figure 2.1 is now known as a multigraph. In general, a **multigraph** is a set of vertices in which some pairs are joined by an edge, or a number of edges. For instance, in the multigraph of Figure 2.1, the vertices A and C are joined by no edge, the vertices B and D are joined by an edge, and the vertices B and C are joined by two edges. A multigraph can conveniently represented by a diagram as shown in Figure 2.1, where vertices are represented by small circles or dots, and edges by line segments or curves. The relative positions of vertices and the lengths of line segments or curves are immaterial. Only the linking relations among the vertices and the number of links joining two vertices that count. Thus the situation for the Konigsberg Bridge Problem can equally well be represented by the multigraph of Figure 2.2

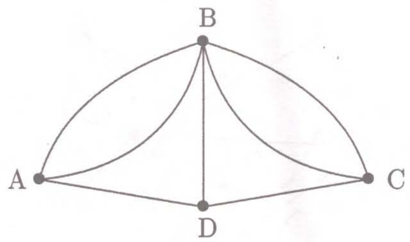


Figure 2.2

More examples of multigraphs are shown in Figure 2.3.

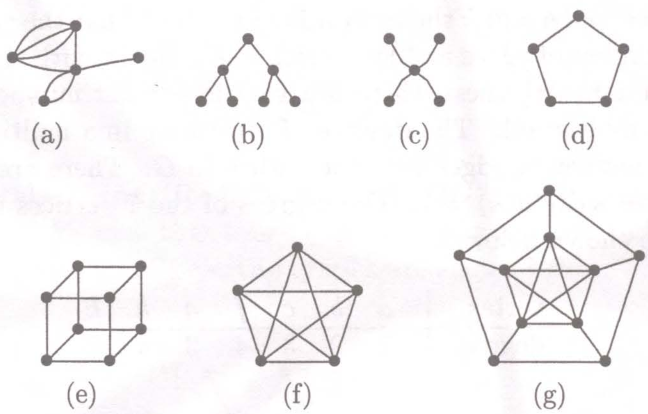


Figure 2.3

Among the multigraphs shown in Figure 2.3, only the multigraph (a) has the feature that some pairs of vertices are joined by at least two edges. In each of the multigraphs (b)-(g), every two vertices are joined by at most one edge. We also call these multigraphs (b)-(g) **simple graphs** or, simply, **graphs**. Thus every graph is a multigraph, but not conversely. Note that no edge is allowed to join a vertex to itself.

Let us proceed to learn some basic terms on multigraphs that will be found useful later. For a multigraph G, let $V(G)$ denote the set of its vertices and $E(G)$ the set of its edges. Two vertices in G are said to be **adjacent** if they are joined by an edge in G. A vertex u is said to be **incident with** an edge e in G if u is joined by e to some vertex. As an example, consider the multigraph G of Figure 2.4. It has 7 vertices and 12 edges, and

$$V(G) = \{a, b, c, f, g, h, k\}, \quad E(G) = \{e_1, e_2, \cdots, e_{12}\}.$$

G:

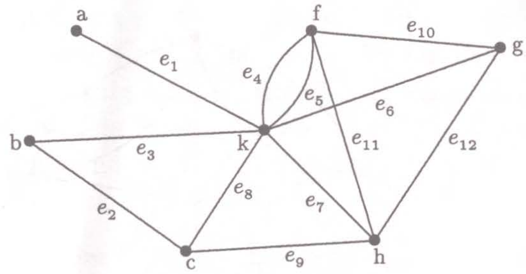


Figure 2.4

The vertices a and k are adjacent, but a and b are not. The vertex a is incident with the edge e_1 but not e_3. The vertices a and k are joined by the edge e_1. We may write $e_1 = ak$, and call a and k the two **ends** of e_1. Note that the vertex a is incident with the least number of edges and the vertex k is incident with the most number of edges. In connection with these, there is a very important number associated with each vertex in a multigraph. The **degree** of a vertex v in a multigraph G, denoted by $d(v)$, is the number of edges incident with v in G. There are 7 edges incident with k, and so we write $d(k) = 7$. The degrees of the 7 vertices in the multigraph of Figure 2.4 are shown below.

vertex	a	b	c	f	g	h	k
degree	1	2	3	4	3	4	7

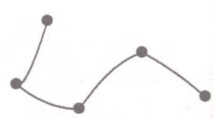
3. Special Families of Graphs

In this section, we confine ourselves to graphs and introduce some important families of graphs.

(1) P_n: the **path** with n vertices, $n \geq 2$.

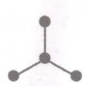
P_3:

P_5:

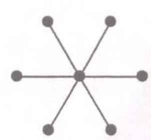


(2) S_n: the **star** with n vertices, $n \geq 4$.

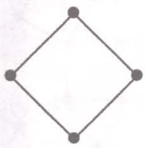
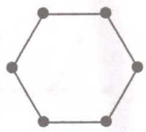
S_4:



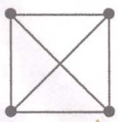
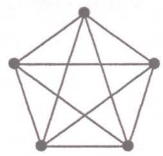
S_7:



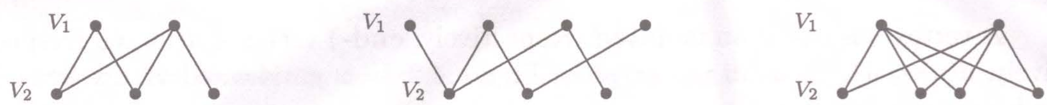
(3) C_n: the **cycle** with n vertices, $n \geq 3$.

C_4:  C_6: 

(4) K_n: the **complete** graph with n vertices, $n \geq 2$ (a graph in which every 2 vertices are adjacent).

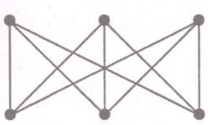
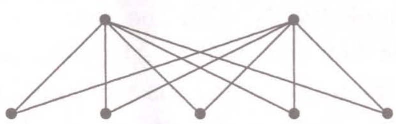
K_4:  K_5: 

(5) A graph G is called **bipartite** if its vertex set $V(G)$ is the union of two nonempty disjoint sets V_1 and V_2 such that every edge of G has one end in V_1 and the other end in V_2. Some examples of bipartite graphs are shown below.



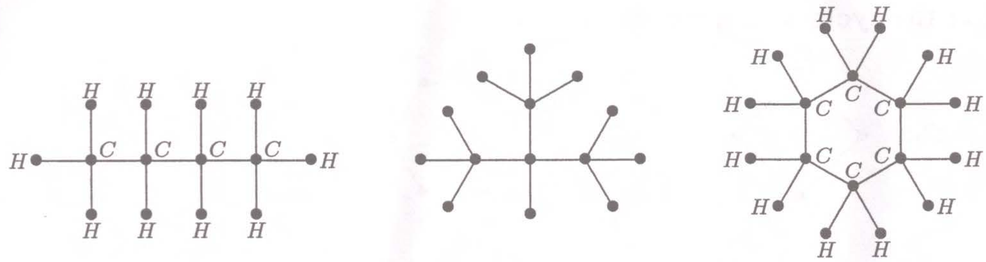
H_1 H_2 H_3

Note that the bipartite graph H_3 has an additional feature that every vertex in V_1 is adjacent to every vertex in V_2. We call H_3 a **complete bipartite** graph and denote it by $K(2,4)$ as there are 2 vertices in V_1 and 4 vertices in V_2. Some other examples of complete bipartite graphs are shown below.

$K(3,3)$:  $K(2,5)$: 

4. Degrees

Recall that the degree $d(v)$ of a vertex v in a multigraph is the number of edges incident with v. The degree of v is also called the **valency** of v as it is related to the valency of an atom in chemical compounds as shown in Figure 4.1.



Butane and Isobutane, C_4H_{10} Cyclohexane, C_6H_{12}

Figure 4.1

Exercise 1. Find the degree of each vertex in the following graph:

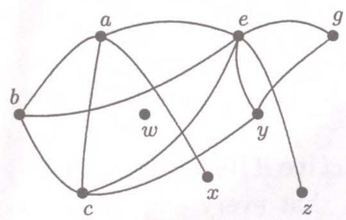


Figure 4.2

A vertex v is called an **isolated** (respectively, **end-**) vertex if $d(v) = 0$ (respectively, $d(v) = 1$). Thus in the graph of Figure 4.2, w is an isolated vertex while x and z are end-vertices.

Exercise 2. A group of 7 people attended a party and various handshakes took place. No one shook hands with the same person more than once, and no one shook his/her own hand. Are there at least two people in the group who had the same number of handshakes?

Exercise 3. Mr. and Mrs. Tan attended a party at which there were 3 other couples. Various handshakes took place. No one shook hands with his/her spouse, no one shook hands with the same person more than once, and of course no one shook his/her own hand. After all the handshaking were finished, Mr. Tan asked each person, including his wife, how many hands he or she had shaken. To his surprise, each gave a different answer. How many hands did Mrs. Tan shake?

Exercise 4. Consider the graph G of Figure 4.2. What is the sum

$$d(a) + d(b) + d(c) + d(e) + d(g) + d(w) + d(x) + d(y) + d(z)?$$

How many edges are there in G? Is there any relation between these two answers?

A vertex v in a multigraph is said to be **even** (respectively, **odd**) if $d(v)$ is even (respectively, odd).

Exercise 5. In the graph of Figure 4.2, how many odd vertices are there? Can you construct a multigraph which contains exactly 1 odd vertex (respectively, exactly 3 odd vertices)?

In Exercise 4, the total sum of the degrees of the vertices is 24 and there are 12 edges in the graph. Notice that $24 = 2 \cdot 12$. In general, is it true that the total sum of the degrees of the vertices in any multigraph is double the number of edges in that multigraph? Yes! It is true in general. Indeed, to count the total sum, we count each edge twice, once for each end of the edge. This simple but useful observation, known as Euler's Handshaking Lemma (1736), is stated below.

Euler's Handshaking Lemma *Let G be a multigraph with $V(G) = \{v_1, v_2, \cdots, v_n\}$ and containing exactly m edges. Then*

$$\sum_{i=1}^{n} d(v_i) = 2m.$$

In Exercise 5, you would not be able to construct a multigraph having exactly 1 odd vertex (respectively, 3 odd vertices). Why? We leave it to you to deduce the following observation from the above lemma.

Corollary *The number of odd vertices in any multigraph is always even.*

Exercise 6. A graph G has 8 vertices and 15 edges. Every vertex in G is of degree 3 or 5. How many vertices of degree 5 does G have? Construct one such graph G.

Exercise 7. A graph H has 10 vertices. The degree of each vertex is between 3 and 5 inclusive. Not every vertex is even. No two odd vertices are of the same degree. How many edges are there in H?

A graph is said to be **regular** if every vertex of the graph has the same degree. More precisely, a graph G is said to be k-**regular** if $d(v) = k$ for any vertex v in G. Two regular graphs are shown in Figure 4.3. The C_5 is 2-regular while the other is 3-regular.

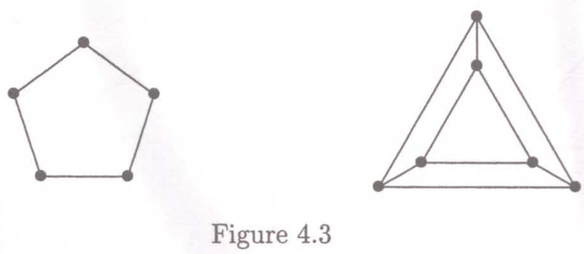


Figure 4.3

Exercise 8. Which graphs introduced in Section 3 are regular?

Exercise 9. Suppose G is a k-regular graph with n vertices and m edges. Find a relation among k, n and m.

Does there exist a 3-regular graph with eight vertices?
Does there exist a 3-regular graph with nine vertices?