COMPETITION CORNER

In this issue we publish the problems of 43[rd] Czech (and Slovak) Mathematical Olympiad, 1994, Ukrainian Mathematical Olympiad, 2002 and 43[rd] International Mathematical Olympiad held in Glasgow, United Kingdom, July 2002.

Please send your solutions of these Olympiads to me at the address given. All correct solutions will be acknowledged. We also present solutions of Canadian Mathematical Olympiad 1993, as well the problems used to select the Singapore Team to the 2002 International Mathematical Olumpiad.

All communications about this column should be sent to :

Dr. Tay Tiong Seng,
Department of Mathematics,
National University of Singapore,
2 Science Drive 2,
Singapore 117543
or by e-mail to mattayts@nus.edu.sg.

• Problems •

43rd Czech (and Slovak) Mathematical Olympiad, 1994

1. Let $\mathbb{N}$ be the set of all natural numbers and $f : \mathbb{N} \to \mathbb{N}$ a function which satisfies the inequality

$$f(x) + f(x+2) \leq 2f(x+1) \quad \text{for any } x \in \mathbb{N}.$$

Prove that there exists a line in the plane which contains infinitely many points with coordinates $(n, f(n))$.

2. A cube of volume V contains a convex polyhedron M. The perpendicular projection of M into each face of the cube coincides with all of this face. What is the smallest possible volume of the polyhedron M?

3. A convex 1994-gon M is drawn in the plane together with 997 of its diagonals drawn. Each diagonal divides M into two sides. The number of edges on the shorter side is defined to be the length of the diagonal. Is it possible to have

(a) 991 diagonals of length 3 and 6 of length 2?

(b) 985 diagonals of length 6, 4 of length 8 and 8 of length 3?

4. Let $a_1, a_2, \ldots$ be an arbitrary sequence of natural numbers such that for each n, the number $(a_n - 1)(a_n - 2)\ldots(a_n - n^2)$ is a positive integral multiple of n^{n^2-1}. Prove that for any finite set P of prime numbers, the following inequality holds:

$$\sum_{p \in P} \frac{1}{\log_p a_p} < 1.$$

5. Let AA_1, BB_1, CC_1 be the heights of an acute-angles triangle ABC (i.e., A_1 lies on the line BC and $AA_1 \perp BC$, etc.) and V their intersection. If the triangles AC_1V, BA_1V, CB_1V have the same areas, does it follow that the triangle ABC is equilateral?

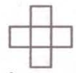
6. Show that from any quadruple of mutually different numbers lying in the interval $(0,1)$ it's possible to choose two numbers $a \neq b$ in such a way that

$$\sqrt{(1-a^2)(1-b^2)} > \frac{a}{2b} + \frac{b}{2a} - ab - \frac{1}{8ab}.$$

Ukrainian Mathematical Olympiad, 2002

Selected problems.

1. (9th grade) The set of numbers $1, 2, \ldots, 2002$ is divided into 2 groups, one comprising numbers with odd sums of digits and the other comprising numbers with even sums odd digits. Let A be the sum of the numbers in the first group and B be the sum of the numbers in the second group. Find $A - B$.

2. (9th grade) What is the minimum number of the figure  that we may mark on the cells of the (8×8) chessboard so that it's impossible to mark more such figures without overlapping?

3. (10th grade) Let A_1, B_1, C_1 be the midpoints of arcs BC, CA, AB of the circumcircle of $\triangle ABC$, respectively. Let A_2, B_2, C_2 be the tangency points of the incircle of $\triangle ABC$, with sides BC, CA, AB, respectively Prove that the lines A_1A_2, B_1B_2, C_1C_2 are concurrent.

4. (10th grade) Find the largest K such that the inequality

$$\frac{1}{(x+y+z)^2} + \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} \geq \frac{K}{\sqrt{(x+y+z)xyz}}$$

holds for all positive x, y, z.

5. (11th grade) Solve in integers the following equation

$$n^{2002} = m(m+n)(m+2n)\cdots(m+2001n).$$

6. (11th grade) Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(x)f(x+y) + 2f(x+2y) + f(2x+y)f(y) = x^4 + y^4 + x^2 + y^2.$$

7. (11th grade) Let C_1, A_1, B_1 be the points at the sides of a given acute $\triangle ABC$ such that $A_1B = A_1C_1$, $A_1C = A_1B_1$. Let I_1 be the incentre of $\triangle A_1B_1C_1$ and H be the orthocentre of $\triangle ABC$. Prove that the points B_1, C_1, I_1, H are concyclic.

8. (11th grade) Let $a_1, a_2, \ldots, a_n$, $n \geq 1$, be real numbers ≥ 1 and $A = 1 + a_1 + \cdots + a_n$. Define x_k, $0 \leq k \leq n$ by

$$x_0 = 1, \quad x_k = \frac{1}{1 + a_k x_{k-1}}, \; 1 \leq k \leq n.$$

Prove that

$$x_1 + x_2 + \cdots + x_n > \frac{n^2 A}{n^2 + A^2}.$$

43rd International Mathematical Olympiad

Glasgow, United Kingdom, July 2002

1. Let n be a positive integer. let T be the set of points (x, y) in the plane where x and y are non-negative integers and $x + y < n$. Each point of T is coloured red or blue. If a point (x, y) is red, then so are all points (x', y') of T with both $x' \le x$ and $y' \le y$. Define an X-set to be a set of n blue points having distinct x-coordinates, and a Y-set to be a set of n blue points have distinct y-coordinates. Prove that the number of X-sets is equal to the number of Y-sets.

2. Let BC be a diameter of the circle Γ with centre O. Let A be a point on Γ such that $0° < \angle AOB < 120°$. Let D be the midpoint of the arc AB not containing C. The line through O parallel to DA meets the line AC at J. The perpendicular bisector of OA meets Γ at E and at F. prove that J is the incentre of the triangle CEF.

3. Find all pairs of integers $m, n \ge 3$ such that there exist infinitely many positive integers a for which

$$\frac{a^m + a - 1}{a^n + a^2 - 1}$$

is an integer.

4. Let n be an integer greater than 1. The positive divisors of n are $d_1, d_2, \ldots, d_k$, where

$$1 = d_1 < d_2 < \cdots < d_k = n.$$

Define $D = d_1 d_2 + d_2 d_3 + \cdots + d_{k-1} d_k$.

(a) Prove that $D < n^2$.

(b) Determine all n for which D is a divisor of n^2.

5. Find all functions f from the set $\mathbb{R}$ of real numbers to itself such that

$$\big(f(x) + f(z)\big)\big(f(y) + f(t)\big) = f(xy - zt) + f(xt + yz)$$

for all $x, y, z, t \in \mathbb{R}$.

6. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be circles of radius 1 in the plane, where $n \ge 3$. Denote their centres by $O_1, O_2, \ldots, O_n$, respectively. Suppose that no line meets more than two of the circles. Prove that

$$\sum_{1 \le i < j \le n} \frac{1}{O_i O_j} \le \frac{(n-1)\pi}{4}.$$

• Solutions •

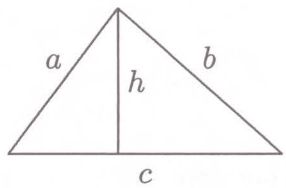
Canadian Mathematical Olympiad 1993

1. Determine a triangle whose three sides and an altitude are four consecutive integers and for which this altitude partitions the triangle into two right triangles with integer sides. Show that there is only one such triangle.

Solved by Zhao Yan (Raffles Institution), Charmaine Sia (Raffles Girls' School) and Colin Tan Weiyu (Raffles Junior College). We present Tan's solution.

Let the sides of the triangle be a, b, c and the altitude to the side c be h (see figure). Then $a, b, c \leq h+3$ and

$$h-3 \leq c = \sqrt{a^2-h^2}+\sqrt{b^2-h^2} \leq 2\sqrt{(h+3)^2-h^2}.$$



Squaring and simplifying, we get

$$h^2 - 18h - 4 \leq 0, \quad \text{or} \quad h \leq 18.$$

The only Pythagorean triples (p, q, r) with $\min\{p, q\} \leq 18$ and $r \leq 21$ are:

$$(3,4,5), \quad (6,8,10), \quad (9,12,15), \quad (12,16,20), \quad (5,12,13), \quad (8,15,17).$$

The altitude h belongs to 2 different such triples. Thus $h = 12$, $a = 13$, $b = 15$, $c = 14$.

2. Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence

$$x,\ x+1,\ x+2,\ x+3, \cdots.$$

Solved by Charmaine Sia (Raffles Girls' School) and Colin Tan Weiyu (Raffles Junior College). Their solutions are similar. Without loss of generality, we can assume that x is the first of the required three distinct terms. First we suppose that there exist integers $m < n$ such that $x, x+m, x+n$ form a geometric progression. Then it follows from

$$\frac{x+m}{x} = \frac{x+n}{x+m}$$

that $x = \frac{m^2}{n-2m}$ which is rational.

Conversely, suppose $x = p/q$ is rational with p, q integers. Let $m = p$ and $n = pq + 2p$. Then $x, x+m, x+n$ form a geometric progression with common ratio $q+1$.

3. In triangle ABC, the medians to the sides AB and AC are perpendicular. Prove that $\cot B + \cot C \geq \frac{2}{3}$.

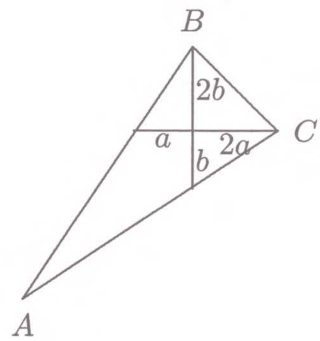
Solved by Charmaine Sia (Raffles Girls' School) and Colin Tan Weiyu (Raffles Junior College). Their solutions are similar. Note that the intersection of the two medians is the centroid M. Let $\angle ABM = \beta_1, \angle CBM = \beta_2, \angle BCM = \alpha_2, \angle ACM = \alpha_1$, With the notation in the figure, we have:

$$\tan\beta_1 = \frac{a}{2b}, \quad \tan\beta_2 = \frac{a}{b}, \qquad \tan\gamma_1 = \frac{b}{2a}, \quad \tan\gamma_2 = \frac{b}{a}.$$

Thus

$$\tan B = \frac{\tan\beta_1 + \tan\beta_2}{1 - \tan\beta_1\tan\beta_2} = \frac{3ab}{2b^2 - a^2},$$

$$\tan C = \frac{3ab}{2a^2 - b^2}.$$



Therefore

$$\cot B + \cot C = \frac{a^2 + b^2}{3ab} \geq \frac{2ab}{3ab} = \frac{2}{3}.$$

4. A number of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a *single* and that between a boy and a girl was called a *mixed single*. The total number boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most how many schools were represented by an odd number of players?

Solved by Calvin Lin Ziwei (Hwachong Junior College) and Charmaine Sia (Raffles Girls' School). We present Calvin Lin's solution. Let b_i and g_i denote the number of boys and girls, respectively, from the i^{th} school. Suppose s and m, respectively, denote the total number of singles and mixed singles played. Further let $d_i = b_i - g_i$. Then

From the given condition, we have

$$\left|\sum_i d_i\right| \le 1, \quad \left|\sum_{i<j} d_i d_j\right| \le 1.$$

Since

$$\sum_i d_i^2 = \Big(\sum_i d_i\Big)^2 - 2\sum_{i<j} d_i d_j \le 3$$

it follows that at most 3 values of d_i can be different from 0. Since $b_i + g_i$ is odd if and only if d_i is odd, the maximum number of schools with an odd number of participants is 3. It is easy to check that a tournament with 3 schools only, two with only 1 girl player and the last with only 1 boy player, satisfy the required conditions. Thus the answer is 3.

5. Let $y_1, y_2, y_3, \ldots$ be a sequence such that $y_1 = 1$ and, for $k > 0$, is defined by the relationship:

$$y_{2k} = \begin{cases} 2y_k & \text{if } k \text{ is even} \\ 2y_k + 1 & \text{if } k \text{ is odd} \end{cases}$$

$$y_{2k+1} = \begin{cases} 2y_k & \text{if } k \text{ is odd} \\ 2y_k + 1 & \text{if } k \text{ is even.} \end{cases}$$

Show that the sequence $y_1, y_2, y_3, \ldots$ takes on every positive integer value exactly once.

We present the solution by Gideon Tan (Raffles Junior College). Similar solutions are also obtained by Charmaine Sia (Raffles Girls' School), Colin Tan Weiyu (Raffles Junior College) and Zhao Yan (Raffles Institution).

We shall prove by induction that

$$\{y_i : i = 2^n, \ldots, 2^{n+1} - 1\} = \{2^n, \ldots, 2^{n+1} - 1\}.$$

It is trivially true for $n = 1$. Now we assume that it's true for some $n \ge 1$. Consider the case $n + 1$. We first note that:

$$\max\{y_i : i = 2^{n+1}, \ldots, 2^{n+2} - 1\} = 2^{n+2} - 1,$$
$$\min\{y_i : i = 2^{n+1}, \ldots, 2^{n+2} - 1\} = 2^{n+1}.$$

Next we note that $y_i \ne y_j$ if $2^{n+1} \le i < j \le 2^{n+2} - 1$. Therefore

$$\{y_i : i = 2^{n+1}, \ldots, 2^{n+2} - 1\} = \{2^{n+1}, \ldots, 2^{n+2} - 1\}.$$

This completes the proof by induction.

Singapore IMO National Team Selection Tests 2002

Note: Problems 1 to 6 were used to select the national team trainees. The last 11 problems were used to select the final team of 6 to represent Singapore at the International Mathematical Olympiad to be held at Glasgow, United Kingdom in July 2002.

1. Let A, B, C, D, E be five distinct points on a circle Γ in the clockwise order and let the extensions of CD and AE meet at a point Y outside Γ. Suppose X is a point on the extension of AC such that XB is tangent to Γ at B. Prove that $XY = XB$ if and only if XY is parallel DE.

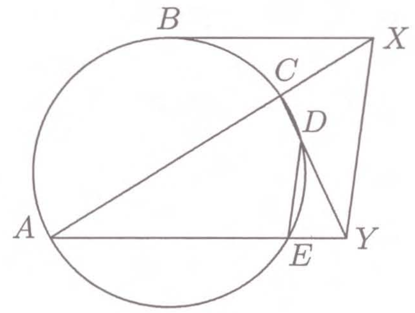
Solution. Suppose $XY = XB$. Then

$$XY^2 = XB^2 = XC \cdot XA$$

so that $XY : XC = XA : XY$. This shows that $\triangle XCY \simeq \triangle XYA$. Hence

$$\angle EDY = \angle XAY = \angle XYC.$$

Therefore, XY is parallel to DE. The converse is similar.



2. Let n be a positive integer and $(x_1, x_2, \ldots, x_{2n})$, $x_i = \pm 1$, $i = 1, 2, \ldots, 2n$ be a sequence of $2n$ integers. Let S_n be the sum

$$S_n = x_1x_2 + x_3x_4 + \cdots + x_{2n-1}x_{2n}.$$

If O_n is the number of sequences such that S_n is odd and E_n is the number of sequences such that S_n is even, prove that

$$\frac{O_n}{E_n} = \frac{2^n - 1}{2^n + 1}.$$

Solution. We can prove by induction that $O_n = 2^{2n-1} - 2^{n-1}$ and $E_n = 2^{2n-1} + 2^{n-1}$. We merely have to note that

$$O_{n+1} = E_n + 3O_n, \quad E_{n+1} = 3E_n + O_n.$$

3. For every positive integer n, show that there is a positive integer k such that

$$2k^2 + 2001k + 3 \equiv 0 \pmod{2^n}.$$

Solution. We show more generally that $ak^2 + bk + c \equiv 0 \pmod{2^n}$ has a solution for all n whenever b is odd and a or c is even. For $n = 1$, take $k = 0$ if c is even and $k = 1$ if c is odd. Now suppose the claim is true for all n. If c is even, then, by assumption, the congruence $2at^2+bt+c/2 \equiv 0 \pmod{2^n}$ has some solution t. Letting $k=2t$ we get $ak^2+bk+c = 2(2at^2+bt+c/2) \equiv 0 \pmod{2^{n+1}}$. If c is odd, then a is even, so $a+b+c$ is even; hence, by assumption, the congruence $2at^2+(2a+b)t+(a+b+c)/2 \equiv 0 \pmod{2^n}$ has some solution t. Letting $k=2t+1$ yields

$$ak^2+bk+c = 2[2at^2+(2a+b)t+(a+b+c)/2] \equiv 0 \pmod{2^{n+1}}.$$

Thus, whether c is even or odd, the claim is true for $n+1$, and so by induction for all n.

4. Let x_1, x_2, x_3 be positive real numbers. Prove that

$$\frac{(x_1^2+x_2^2+x_3^2)^3}{(x_1^3+x_2^3+x_3^3)^2} \le 3.$$

Solution. Consider the function $f(x) = x^{3/2}$ for $x>0$. $f''(x) = \frac{3}{\sqrt{x}} > 0$ for $x>0$. Hence, f is concave upward. By Jensen's Inequality, for any three positive numbers z_1, z_2, z_3,

$$f\left(\frac{z_1+z_2+z_3}{3}\right) \le \frac{f(z_1)+f(z_2)+f(z_3)}{3}.$$

Now take $z_1 = x_1^2$, $z_2 = x_2^2$ and $z_3 = x_3^2$. We have

$$\left(\frac{x_1^2+x_2^2+x_3^2}{3}\right)^{\frac{3}{2}} \le \frac{x_1^3+x_2^3+x_3^3}{3}.$$

That is

$$\frac{(x_1^2+x_2^2+x_3^2)^3}{(x_1^3+x_2^3+x_3^3)^2} \le 3.$$

5. For each real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x. For example $\lfloor 2.8 \rfloor = 2$. Let $r \ge 0$ be a real number such that for all integers m, n, $m|n$ implies $\lfloor mr \rfloor | \lfloor nr \rfloor$. Prove that r is an integer.

Solution. Suppose r is not an integer, choose an integer a such that $\lfloor ar \rfloor$ is an integer greater than 1 while ar itself is not an integer. Let k be the unique integer such that

$$\frac{1}{k+1} \le ar - \lfloor ar \rfloor < \frac{1}{k}.$$

Then

$$1 \le (k+1)(ar - \lfloor ar \rfloor) < \frac{k+1}{k} \le 2.$$

Since

$$\lfloor (k+1)ar \rfloor = (k+1)\lfloor ar \rfloor + \lfloor (k+1)(ar - \lfloor ar \rfloor) \rfloor = (k+1)\lfloor ar \rfloor + 1$$

we see that $\lfloor ar \rfloor$ does not divide $\lfloor (k+1)ar \rfloor$. Thus $m = a$, $n = (k+1)a$ form a counter example.

6. Find all functions $f : [0, \infty) \longrightarrow [0, \infty)$ such that $f(f(x)) + f(x) = 12x$, for all $x \geq 0$.

Solution. Fix any $x \geq 0$. Let $f^{[0]}(x) = x$ and $f^{[1]}(x) = f(x)$. For $n \geq 1$, let $f^{[n]}(x) = f(f^{[n-1]}(x))$. Then the above functional equation gives

$$f^{[n+2]}(x) + f^{[n+1]}(x) = 12 f^{[n]}(x).$$

Solving this difference equation, we have

$$f^{[n]}(x) = C_1 3^n + C_2(-4)^n.$$

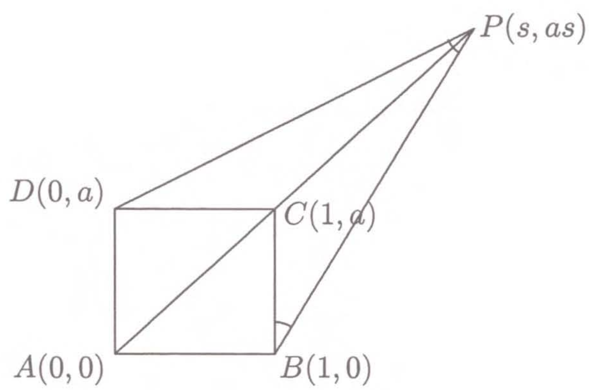
Using the initial conditions $f^{[0]}(x) = x$ and $f^{[1]}(x) = f(x)$, we have $C_1 = (f(x)+4)/7$ and $C_2 = (3x - f(x))/7$. Therefore,

$$f^{[n]}(x) = \frac{1}{7}(f(x)+4)3^n + \frac{1}{7}(3x - f(x))(-4)^n.$$

Since $f(x) \geq 0$, $f^{[n]}(x) \geq 0$ for all $n \geq 0$. By taking n to be even, we have $\frac{1}{7}(f(x)+4)3^n + \frac{1}{7}(3x - f(x))4^n \geq 0$. From this, $3x - f(x) \geq 0$. By taking n to be odd, we have $\frac{1}{7}(f(x)+4)3^n - \frac{1}{7}(3x - f(x))4^n \geq 0$. From this, $3x - f(x) \leq 0$. Consequently, $f(x) = 3x$. One can easily verify that $f(x) = 3x$ satisfies the given functional equation.

7. Let P a point on the extension of the diagonal AC of rectangle $ABCD$ beyond C, such that $\angle BPD = \angle CBP$. Determine the value of $\frac{PB}{PC}$.

Solution . Set up a coordinate system as shown in the diagram below, where $s > 1$



Let

$$k = \frac{PB^2}{PC^2} = \frac{a^2s^2 + (s-1)^2}{(as-a)^2 + (s-1)^2}.$$

Simplifying, we have

$$a^2[(k-1)s^2 + k - 2ks] + (k-1)(s-1)^2 = 0. \quad (1)$$

Now we make use of the condition that $\angle BPD = \angle CBP$. In other words,

$$\frac{\overrightarrow{BC} \cdot \overrightarrow{BP}}{|\overrightarrow{BC}||\overrightarrow{BP}|} = \frac{\overrightarrow{PD} \cdot \overrightarrow{PB}}{|\overrightarrow{PD}||\overrightarrow{PB}|}$$

That is,

$$\frac{\langle 0, a\rangle \cdot \langle s-1, as\rangle}{a} = \frac{\langle s-1, as\rangle \cdot \langle s, as-a\rangle}{\sqrt{s^2 + (as-a)^2}}.$$

Simplifying, we have $a^2(s^2 - 4s + 2) + (s-1)^2 = 0$. Combining this with (1), we have $(k-2)(2s-1) = 0$. Since $s > 1$, we see that $k = 2$.

8. Let $a_1, a_2 \ldots, a_n$ be positive real numbers and let $A = \sum a_i$. Prove that

$$\sum \frac{a_i}{2A - a_i} \geq \frac{n}{2n-1}.$$

Solution. (Meng Dazhe (Raffles Junior College)) Let $x_i = 2A - a_i$. Then $x_i > 0$ for all i. Hence, using $AM \geq HM$, we have

$$\begin{aligned}\sum \frac{a_i}{2A - a_i} &= \sum \frac{2A - x_i}{x_i} = 2A \sum \left(\frac{1}{x_i}\right) - n \\ &\geq \frac{2An^2}{x_1 + x_2 + \cdots + x_n} - n = \frac{2An^2}{(2n-1)A} - n = \frac{n}{2n-1}.\end{aligned}$$

9. Suppose the sum of m pairwise distinct positive even numbers and n pairwise distinct positive odd numbers is 2002. What is the maximum value of $3m + 4n$?

Solution. Let $a_1 > a_2 > \cdots > a_m$ be the even numbers and $b_1 > b_2 > \cdots > b_n$ be the odd numbers. Then

$$\begin{aligned}\textstyle\sum a_i &\geq 2 + 4 + \cdots + 2m = m(m+1) \\ \textstyle\sum b_i &\geq 1 + 3 + \cdots + (2n-1) = n^2 \\ 2002 &= \textstyle\sum a_i + \sum b_i \geq m(m+1) + n^2.\end{aligned}$$

Therefore

$$n \leq \sqrt{2002 - m(m+1)}.$$

Thus

$$3m + 4n \leq 4\sqrt{2002 - m(m+1)} + 3m.$$

Write $y = 3m + 4n$, we get

$$25m^2 - (6y - 16)m + y^2 - 2002 \times 16 \leq 0$$

The discriminant

$$\Delta = (6y - 16)^2 - 100(y^2 - 2002 \times 16) \geq 0.$$

Hence

$$y^2 + 3y - 50054 \leq 0.$$

Note that $y > 0$. Thus

$$0 < y \leq \frac{-3 + 5\sqrt{8009}}{2} < 222.3$$

So $\max y = 222$.

This maximum value of $3m + 4n$ can be easily achieved by taking for example 26 even numbers : $2, 4, 6, \ldots, 48, 50, 56$ and 36 odd numbers $1, 3, 5, \ldots, 69, 71$. Their sum is 2002 and $3m + 4n$ is exactly 222.

10. Determine the number of positive integers that can be expressed in the form

$$\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{2002}{a_{2002}},$$

where $a_1, a_2 \ldots, a_{2002}$ are positive integers.

Solution 1. One can use induction on k to prove that

$$S = \frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{k}{a_k}$$

can take all integer values in the interval $[1, \binom{k+1}{2}]$.

The assertion is obviously true for $k = 1$ and 2. Suppose the assertion is true when $k = t$.

(1) Take $a_{t+1} = 1$. By induction hypothesis, $\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{t}{a_t}$ can take all integer values in $[1, \binom{t+1}{2}]$. Thus, $\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{1+t}{a_{t+1}}$ can take all integer values in $[t + 2, \binom{t+2}{2}]$.

(2) Take $a_{t+1} = t + 1$. By induction hypothesis, $\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{1+t}{a_{t+1}}$ can take all integer values in $[2, \binom{t+1}{2} + 1]$.

(3) Take $a_i = \binom{t+1}{2}$ for all $i = 1, 2, \ldots t+1$. Then $\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{1+t}{a_{t+1}} = 1$.

Thus, $\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{1+t}{a_{t+1}}$ can take all integer values in $[1, \binom{t+2}{2}]$.

Solution 2. First we assert that any integer between 1 and $\frac{1}{2}k(k+1)$ can be written as a sum of at most k distinct integers among $1, 2, \ldots, k$. This can be proved by induction on k. If $k = 1$ or 2, then we can easily check that the statement is true. Let $k \geq 2$. Consider any integer m between 1 and $\frac{1}{2}(k+1)(k+2)$. If $m < k(k+1)/2$, the result follows from the induction hypothesis. Assuming $k(k+1)/2 < m \leq (k+1)(k+2)/2$. Then $0 < m - (k+1) \leq k(k+1)/2$, Using induction hypothesis, we may write $m - (k+1)$ as a sum of at most k distinct integers among $1, 2, \ldots, k$. Thus $m = m - (k+1) + (k+1)$ can be written as a sum at most $k+1$ distinct integers among $1, 2, \ldots, k+1$.

Alternatively, locate m in $[k(k+1)/2, (k+1)(k+2)/2)$. That is $m = \frac{1}{2}(k+1)(k+2) - i$, where $1 \leq i \leq k+1$. Then we may write $m = 1 + 2 + \cdots + \hat{i} + \cdots + k + (k+1)$, where $\hat{i}$ means the term i is omitted.

Now we prove that

$$S = \frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{2002}{a_{2002}}$$

can take all integer values in the interval $[1, \frac{1}{2}(2002)(2003)] = [1, 2005003]$. Certainly 1 can be written in this form by taking each a_i to be 2005003. Let $m > 1$ be any integer in $[1, 2005003]$. Write $m - 1$ as a sum of at most 2002 distinct integers among $1, 2, \ldots, 2002$. That is $m - 1 = b_1 + \cdots + b_k$, where $1 \leq b_i \leq 2002$ and all are distinct integers. Since $m - 1 < \frac{1}{2}(2002)(2003)$, $k < 2002$. Take $a_j = 1$ if $j \in \{b_1, \ldots, b_k\}$, and $a_j = \frac{1}{2}(2002)(2003) - m + 1$ otherwise. Then $\frac{1}{a_1} + \frac{2}{a_2} + \cdots + \frac{2002}{a_{2002}} = m$.

11. Prove the inequality

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \left(\sum_{i=1}^{n}(a_i + b_i)\right)\left(\sum_{i=1}^{n} \frac{a_i b_i}{a_i + b_i}\right)$$

for any positive numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$.

Solution

$$\text{LHS} = \sum a_i b_i + \sum_{i \neq j} a_i b_j$$

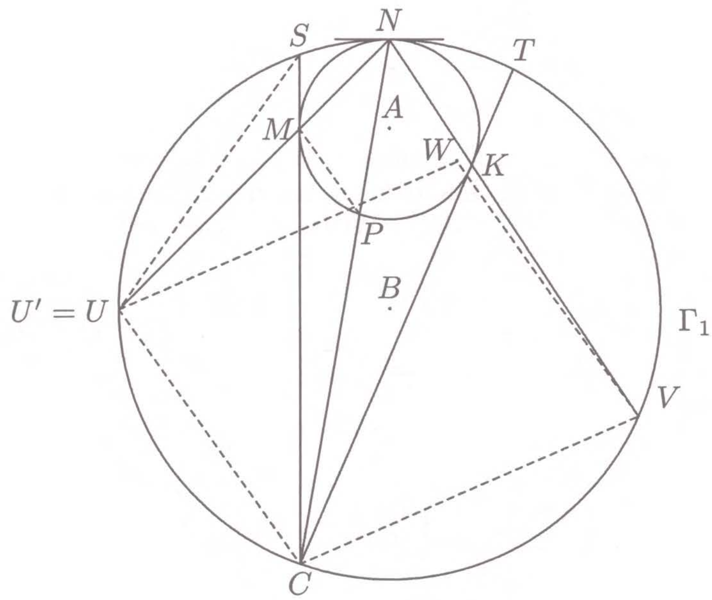
$$\text{RHS} = \sum a_i b_i + \sum_{i \neq j} \frac{a_j b_j (a_i + b_i)}{(a_i + b_i)(a_j + b_j)}.$$

Thus

$$\begin{aligned} \text{LHS} - \text{RHS} &= \sum_{i<j} a_i b_j + a_j b_i - \frac{a_j b_j (a_i + b_i)}{(a_i + b_i)(a_j + b_j)} - \frac{a_i b_i (a_j + b_j)}{(a_i + b_i)(a_j + b_j)} \\ &= \sum_{i<j} \frac{(a_i b_j - a_j b_i)^2}{(a_i + b_i)(a_j + b_j)} \geq 0. \end{aligned}$$

12. Two circles Γ_1 and Γ_2 are tangent to each other internally at a point N such that Γ_2 is inside Γ_1. Points C, S and T are on Γ_1 such that CS and CT are tangent to Γ_2 at M and K respectively. Let U and V be the midpoints of the arcs CS and CT respectively. Prove that $UCVW$ is a parallelogram where W is the second point of intersection between the circumcircles of $\triangle UMC$ and $\triangle VCK$.

Solution. First N, M, U are collinear. To see this, join MP and let the extension of NM meet Γ_1 at U'. Join $U'C$ and $U'S$. Since N is a point of tangency for both circles, we have MP is parallel to $U'C$. Also, $\angle CMP = \angle CNU'$. Therefore, $\angle U'SC = \angle CNU' = \angle CMP = \angle U'CS$. Therefore, U' is the midpoint of the arc SC. That is $U' = U$. Similarly, N, K, V are collinear.

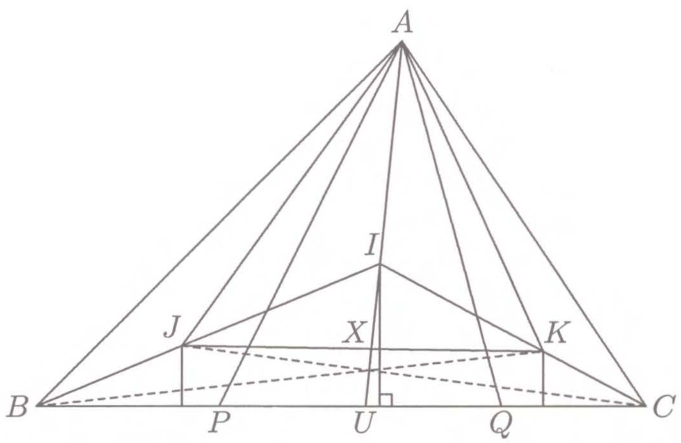


Next we shall show that W lies on MK. Since M, U, C, W are concyclic, $\angle MWC = 180° - \angle MUC$. Similarly, $\angle KWC = 180° - \angle KVC$. Thus, $\angle MWC + \angle KWC = 360° - (\angle MUC + \angle KVC) = 180°$. Therefore, W lies on MK.

Therefore, $\angle UWC = \angle UMC = \angle SMN$. As SM is tangent to Γ_2 at M, we have $\angle SMN = \angle NKM = \angle NKW = \angle WCV$. Thus, $\angle UWC = \angle WCV$. This means that UW is parallel to VC. Similarly, UC is parallel to VW.

13. In triangle ABC, let P and Q lie on the interior of BC such that $\angle BAP = \angle CAQ$. Let I be the incentre of ABC. Also, let J and K be the incentres of triangles BAP and CAQ respectively. Prove that AI, BK and CJ are concurrent.

Solution 1. Let r be the radius of the incircle of ΔABC. Also, let r_b denote the inradius of ΔBAP and let r_c denote the inradius of ΔCAQ. (We can assume that the points B, P, Q and C lie along BC in that order; the case where the order of the points is B, Q, P and C instead is essentially similar.) Note that I and J both lie on the interior angle bisector of $\angle ABC$, and that I and K both lie on the interior angle bisector of $\angle ACB$. By dropping the perpendiculars from I, J and K onto BC, and applying similar triangles to the right-angle triangles produced, we find that $\frac{IB}{BJ} = \frac{r}{r_b}$ and $\frac{IC}{CK} = \frac{r}{r_c}$.



Let the angle bisector of $\angle BAC$ (AI extended) meet the line JK at X. Since $\angle BAP = \angle CAQ$, we have $\angle BAJ = \angle JAP = \angle QAK = \angle KAC$. Also, since $\angle BAI = \angle CAI$, we have $\angle JAI = \angle KAI$. Hence, $\frac{JX}{XK} = \frac{AJ}{AK} = \frac{r_b/\sin BAJ}{r_c/\sin KAC} = \frac{r_b}{r_c}$, since $\angle BAJ = \angle KAC$.

Thus, in ΔIJK, we have

$$\frac{IB}{BJ} \cdot \frac{JX}{XK} \cdot \frac{KC}{CI} = \frac{r}{r_b} \cdot \frac{r_b}{r_c} \cdot \frac{r_c}{r} = 1$$

Hence, by Ceva's Theorem, the lines AI, BK and CJ are concurrent.

Solution 2. Instead of applying Ceva's theorem to $\triangle IJK$, we may apply it to $\triangle IBC$. So we need to show

$$\frac{BU}{UC} \cdot \frac{CK}{KI} \cdot \frac{IJ}{JB} = 1.$$

By angle bisector theorem, $BU/UC = AB/AC$.

Next,

$$\frac{CK}{KI} = \frac{\text{Area } \triangle AKC}{\text{Area } \triangle AIK} = \frac{AK \cdot AC \sin \angle CAK}{AK \cdot AI \sin \angle KAI} = \frac{AC \sin \angle CAK}{AI \sin \angle KAI}.$$

Similarly,

$$\frac{IJ}{BJ} = \frac{AI \sin \angle JAI}{AB \sin \angle BAJ}.$$

As $\angle BAJ = \angle CAK$ and $\angle JAI = \angle KAI$, the result follows.

14. Prove that for any $n \geq 2$ distinct positive integers $a_1, a_2, \ldots, a_n$,

$$\prod_{1 \leq j < i \leq n} \frac{a_i - a_j}{i - j}$$

is an integer.

Solution. Let n, k, s, t be positive integers such that

$$n = ks + t, 0 \leq t < s.$$

Claim 1: The number of terms in $(n-1)!(n-2)! \cdots 1!$ which are multiples of s is given by

$$kt + s\binom{k}{2}.$$

Proof: The number of multiples in $i!$ is $\lfloor i/s \rfloor$. Thus the total number of multiples is

$$\sum_{i=1}^{n-1} \left\lfloor \frac{i}{s} \right\rfloor = s(1 + 2 + \cdots + (k-1)) + kt = kt + s\binom{k}{2}.$$

Let $a_i, 1 \leq i \leq n$ be a strictly increasing sequence of positive integers.

Claim 2: The number of terms in $\prod(a_i - a_j)$ which are multiples of s is at least $kt + s\binom{k}{2}$.

Proof: If r of the terms $a_1, \ldots, a_n$ are pairwise congruent mod s, then, correspondingly, there are $\binom{r}{2}$ terms in the product which are multiples of s. Since for any positive integers, $a < c < d < b$ with $a + b = c + d$, we have $\binom{a}{2} + \binom{b}{2} > \binom{c}{2} + \binom{d}{2}$. Thus the minimum is achieved when t of the congruence classes contain $k + 1$ elements each while the other $s - t$ congruence classes contain k elements each. Thus the minimum is $t\binom{k+1}{2} + (s-t)\binom{k}{2} = s\binom{k}{2} + tk$.

Thus for any prime p, the number of multiples of p^α in the numerator is at least the number of its multiples in the denominator.

[Ps: If $a < b$ are positive integers, we have

$$\binom{a}{2} + \binom{b}{2} \geq \binom{a+1}{2} + \binom{b-1}{2}$$

because

$$\binom{a}{2} + \binom{b}{2} - \binom{a+1}{2} - \binom{b-1}{2} = b - a - 1 \geq 0.]$$

15. A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called *historic* if $\{z - y, y - x\} = \{1819, 1965\}$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.

Solution. Let $a < b$ be the two historic numbers. That is in our case, $a = 1819, b = 1965$.

The sets

$$A = \{0, a, a + b\}, B = \{0, b, a + b\}$$

are historic. Note that a set X is historic if and only if and only if it is either $x + A$ or $x + B$. We'll construct the union as follows:

(1) Take $X_1 = A$.

(2) Suppose $X_1, \ldots X_m$ have been found, X_{m+1} is constructed as follows:

Let k be the smallest integers not in the union U of $X_1, \ldots, X_m$. If $k + a \notin U$, take $X_{m+1} = A + k$. Otherwise, take $X_{m+1} = B + k$.

This construction always works because the smallest element in each of $X_1, \ldots, X_m$ is less than k, thus the largest is less than $k + a + b$. That is $k + a + b \notin U$. So if $k + a \notin U$ then we can take $X_{m+1} = k + A$.

If $k + a \in U$ and $k + b \in U$, then $k + b$ is the largest element is some X_j, $j \leq m$. Thus $k + b = n + a + b$ for some $n < k$. So $k = n + a$. Since $k \notin U$, $X_j = \{n, n + b, n + a + b\}$. But when we chose X_j, we cannot take $n + b$ as the second element, since at that time $n + a$ was still not used. This shows that if $k + a \in U$, then $k + b \notin U$ and we may take $X_{m+1} = B + k$.

16. A set S of nonnegative real numbers is said to be *good* if for any $x, y \in S$, either $x + y \in S$ or $|x - y| \in S$. For example, if r is a positive real number and n is a positive integer, the set $S(n, r) = \{0, r, 2r, \ldots, nr\}$ is *good*. Prove that any finite *good* set which is not the set $\{0\}$ is either of the form $S(n, r)$ or has exactly 4 elements.

Solution. If m is the largest element in the set S, then $m - m = 0 \in S$ as $m + m \notin S$. Thus a good set contains 0. Certainly $\{0\}$ and $\{0, a\}$ are good sets. If $S = \{0, a, b\}$ is a good set with 3 elements, then $b - a \in S$. Hence, $b - a = a$ or $b = 2a$. Thus $S = S(2, a)$.

Let $S = \{0, a, b, c\}$ be a good set with $0 < a < b < c$. Then $S = \{0, c - b, c - a, c\}$ with $0 < c - b < c - a < c$. Thus, $a = c - b$. So a 4-element good set is of the form $S = \{0, a, b, a + b\}$. Such a set is indeed good.

Let $0 < x_1 < x_2 < \cdots < x_n$, $n \geq 4$, be the elements of a good set S with $n+1$ elements. Then for any $i = 1, \ldots, n$, we have $x_n - x_i \in S$. Thus

$$0 < x_n - x_{n-1} < x_n - x_{n-2} < \cdots < x_n - x_2 < x_n - x_1 < x_n$$

are the elements of S. Thus we get $x_n = x_{n-i} + x_i$. Also $(x_n - x_1) - x_i \in S$ for $i = 2, \ldots, n-1$.

Thus

$$0 < x_n - x_1 - x_{n-2} < x_n - x_1 - x_{n-3} < \cdots$$
$$< x_n - x_1 - x_2 < x_n - x_2 < x_n - x_1 < x_n$$

are the elements of S.

Thus we have $x_{i+1} - x_i = x_1$ for $i = 2, \ldots, n-1$. From $x_1 + x_{n-1} = x_2 + x_{n-2}$, we also get $x_{n-1} - x_{n-2} = x_2 - x_1$. Thus $x_2 - x_1 = x_1$ as well. Thus $x_i = ix_1$ and $S = S(n, x_1)$.

17. Find all positive integers n such that $2^n - 1$ is a multiple of 3 and $(2^n - 1)/3$ is a divisor of $4m^2 + 1$ for some integer m.

Solution. We shall show that n is a power of 2. First, observe that $2^n - 1$ is a multiple of 3 if and only if n is even. Let $n = 2^k$. The result is obviously true if $k = 1$. So we consider $k \geq 2$. Then

$$(2^{2^k} - 1)/3 = (2^{2^{k-1}} + 1)(2^{2^{k-2}} + 1) \cdots (2^{2^2} + 1)(2^2 + 1).$$

Note that $F_i = 2^{2^i} + 1$, $i = 2, \ldots, k-1$ are pairwise relatively prime. To see this, consider any positive integers i, j, we have $F_{i+j} - 2 = rF_i$ where r is some integer. If F_i and F_{i+j} has a common divisor d, d must be odd. Also d divides 2. Thus $d = 1$.

Also, $[2^{2^{i-1}}]^2 \equiv -1 \pmod{F_i}$. By Chinese Remainder Theorem, there exists an integer m such that $2m \equiv 2^{2^{i-1}} \pmod{F_i}$ for $i = 2, \ldots, k-1$. Thus, $(2m)^2 \equiv -1 \pmod{F_i}$ for all $i = 2, \ldots, k-1$. In other words, $F_i = 2^{2^i} + 1$ divides $4m^2 + 1$ for all $i = 2, \ldots, k-1$. Therefore, $(2^{2^k} - 1)/3$ divides $4m^2 + 1$.

Now suppose $n = 2^h s$, where s is an odd integer greater than 1. We have $2^n - 1 = (2^s)^{2^h} - 1$. Thus $2^s + 1$ divides $2^n - 1$. Since 3 divides $2^s + 1$ and $2^s + 1 \equiv 1 \pmod 4$, $2^s + 1$ and consequently, $2^n - 1$ has another prime divisor $p \equiv 3 \pmod 4$. If there exists m such that $4m^2 + 1 \equiv 0 \pmod{(2^n - 1)/3}$, then $(2m)^2 \equiv -1 \pmod p$. However, $-1 \equiv (-1)^{(p-1)/2} \equiv ((2m)^2)^{(p-1)/2} \equiv (2m)^{p-1} \equiv 1 \pmod p$, a contradiction. Here the last congruence is by Fermat's Little Theorem and the first congruence is because $p \equiv 3 \pmod 4$.