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Application of the Arithmetic-Geometric Mean Inequality

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Application of the Arithmetic-Geometric Mean Inequality

An easy application of the arithmetic-geometric mean inequality is that for $a > b > 0$,

$$\sqrt{2}a^3 + \frac{3}{ab - b^2} \geq 10, \tag{1}$$

where equality holds if and only if $a = 2b = \sqrt{2}$. Inequality (1) follows from the fact that the arithmetic mean is greater than or equal to the geometric mean. For, $a^2 - 4(ab - b^2) = (a - 2b)^2 \geq 0$ and hence

$$ab - b^2 \leq \frac{a^2}{4}. \tag{2}$$

Thus, $$\sqrt{2}a^3 + \frac{3}{ab - b^2} \geq \sqrt{2}a^3 + \frac{12}{a^2} = \frac{a^3}{\sqrt{2}} + \frac{a^3}{\sqrt{2}} + \frac{4}{a^2} + \frac{4}{a^2} + \frac{4}{a^2}$$
$$\geq 5\sqrt[5]{\left(\frac{a^3}{\sqrt{2}}\right)^2 \left(\frac{4}{a^2}\right)^3} = 10.$$

Hence, from (2), equality in (1) holds if and only if $a = 2b$ and $\frac{a^3}{\sqrt{2}} = \frac{4}{a^2}$, which is equivalent to $a = 2b = \sqrt{2}$. The inequality (1) is generalized as follows.

Proposition 1. *If a, b, c and d are positive real numbers and $a > b$, then*

$$ca^3 + \frac{d}{ab - b^2} \geq 5\sqrt[5]{\frac{16c^2d^3}{27}},$$

where equality holds if and only if $a = 2b = \left(\frac{8d}{3c}\right)^{\frac{1}{5}}$.

Proof. Let $f(x) = cx^3 + \frac{4d}{x^2}$ for $x \in (0, \infty)$. Then $f'(x) = 3cx^2 - \frac{8d}{x^3}$, and $f'(x) = 0$ when $x = \left(\frac{8d}{3c}\right)^{\frac{1}{5}}$. Denote $\left(\frac{8d}{3c}\right)^{\frac{1}{5}}$ by k. Since $\lim_{x\to 0^+} f'(x) = -\infty$, $\lim_{x\to\infty} f'(x) = \infty$, and k is the only critical point of f, the function f is decreasing on the interval $(0, k)$ and increasing on the interval (k, ∞). Hence f has the minimum value at k. Note that $a^2 + 4b^2 - 4ab = (a - 2b)^2 \geq 0$ and hence $ab - b^2 \leq \frac{a^2}{4}$, where equality holds if and only if $a = 2b$. Consequently, $ca^3 + \dfrac{d}{ab-b^2} \geq ca^3 + \dfrac{4d}{a^2} = f(a) \geq f(k) = ck^3 + \dfrac{4d}{k^2} = 5\left(\dfrac{16c^2d^3}{27}\right)^{\frac{1}{5}}$, where equality holds if and only if $a = 2b = \left(\dfrac{8d}{3c}\right)^{\frac{1}{5}}$. Thus the proof is complete.

Corollary 1. *For $a > b > 0$, $\sqrt{2}a^3 + \dfrac{3}{ab-b^2} \geq 10$. Equality holds if and only if $a = 2b = \sqrt{2}$.*

To see this, let $c = \sqrt{2}$ and $d = 3$ in Proposition 1.

Corollary 2. *For $a > b > 0$, $3\sqrt{3}a^3 + \dfrac{\sqrt[3]{2}}{ab-b^2} \geq 10$.*

To see this, let $c = 3\sqrt{3}$ and $d = \sqrt[3]{2}$ in Proposition 1.

Over 200 years ago, Euler introduced a constant, known as Euler's constant and defined by $\lim_{n\to\infty} \gamma_n$, where $\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n$. The inequality $0 < \gamma_n \leq 1$ is widely used to show that $\lim_{n\to\infty} \gamma_n$ exists. This inequality $0 < \gamma_n \leq 1$ is equivalent to the inequality $\ln n < \sum_{k=1}^{n} \frac{1}{k} \leq 1 + \ln n$. Below, better lower bound $n(n+1)^{\frac{1}{n}} - n$ and upper bound $n - (n-1)n^{-\frac{1}{n-1}}$ for $\sum_{k=1}^{n} \frac{1}{k}$ are obtained by applying the arithmetic-geometric mean inequality.

Proposition 2. *For every integer $n > 1$, we have*

$$\ln n < n(n+1)^{\frac{1}{n}} - n < \sum_{k=1}^{n} \frac{1}{k} \leq n - (n-1)n^{-\frac{1}{n-1}} < 1 + \ln n.$$

Proof. By applying the arithmetic-geometric mean inequality, we have,

$$\frac{n + \sum_{k=1}^{n} \frac{1}{k}}{n} = \frac{\sum_{k=1}^{n} \frac{k+1}{k}}{n} > \left(\prod_{k=1}^{n} \frac{k+1}{k}\right)^{\frac{1}{n}} = (n+1)^{\frac{1}{n}},$$

and hence $n(n+1)^{\frac{1}{n}} - n < \sum_{k=1}^{n} \frac{1}{k}$.

Again, by applying the arithmetic-geometric mean inequality, we have

$$\frac{n-\sum_{k=1}^{n}\frac{1}{k}}{n-1}=\frac{\sum_{k=1}^{n}\frac{k-1}{k}}{n-1}=\frac{\sum_{k=2}^{n}\frac{k-1}{k}}{n-1}\geq\left(\prod_{k=2}^{n}\frac{k-1}{k}\right)^{\frac{1}{n-1}}=\left(\frac{1}{n}\right)^{\frac{1}{n-1}},$$

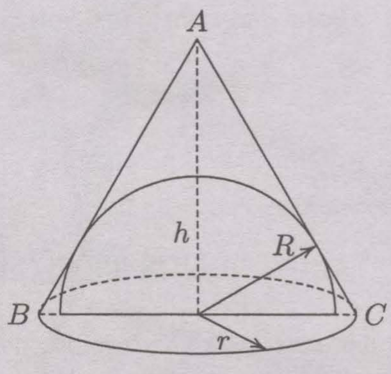
and hence $\sum_{k=1}^{n}\frac{1}{k}\leq n-(n-1)n^{-\frac{1}{n-1}}$.

To prove the other inequality $\ln n < n(n+1)^{\frac{1}{n}} - n$, consider the well-known easy fact that $e^x \geq 1+x$ for all x and $f(x)=e^x$ is an increasing function. Since $e^{(n+1)^{\frac{1}{n}}-1} \geq 1+(n+1)^{\frac{1}{n}}-1 > n^{\frac{1}{n}}$ and $e^{\frac{\ln n}{n}} = n^{\frac{1}{n}}$, we have $e^{(n+1)^{\frac{1}{n}}-1} > e^{\frac{\ln n}{n}}$ and hence $\frac{\ln n}{n} < (n+1)^{\frac{1}{n}}-1$. This shows that $\ln n < n(n+1)^{\frac{1}{n}}-n$. The last inequality of the Proposition 2 is equivalent to $(n-1)(1-n^{-\frac{1}{n-1}}) < \ln n$. Since $e^x > 1+x$ for all nonzero x, we have $1-e^x < -x$ and hence $(1-e^{-\frac{\ln n}{n-1}}) < \frac{\ln n}{n-1}$ and hence $(n-1)(1-n^{-\frac{1}{n-1}}) = (n-1)(1-e^{-\frac{\ln n}{n-1}}) < (n-1)\frac{\ln n}{n-1} = \ln n$. Thus the proof is complete.

The last application of the arithmetic-geometric mean inequality is given below.

Proposition 3. *Let V be the volume of a right circular cone of radius r and height h. Let A be the vertex of the cone and let the triangle ABC be the vertical cross section of the cone. A semicircle of radius R is drawn inside the triangle ABC such that sides AB and AC are tangent to the semicircle and the center of the semicircle is at the center of the base of the cone. Then $V \geq \frac{\sqrt{3}}{2}\pi R^3$. Equality holds if and only if $h=\sqrt{2}r$.*

Proof. Using the properties of similar triangles, one can easily see that $\frac{r}{R}=\frac{\sqrt{h^2+r^2}}{h}$. This implies that $h=\frac{rR}{\sqrt{r^2-R^2}}$. By applying the arithmetic-geometric mean inequality, we have



$$\begin{aligned}4r^6+27R^6 &= 4r^6+\tfrac{27}{2}R^6+\tfrac{27}{2}R^6\\ &\geq 3\sqrt[3]{4r^6\cdot\tfrac{27}{2}R^6\cdot\tfrac{27}{2}R^6}\\ &= 27r^2R^4;\end{aligned}$$

equality holds if and only if $4r^6=\frac{27}{2}R^6$, that is $h=\sqrt{2}r=\sqrt{3}R$. Hence $4r^6 \geq 27R^4(r^2-R^2)$. Consequently, since $V=\frac{1}{3}\pi r^2h$, $h=\frac{rR}{\sqrt{r^2-R^2}}$ and $r^3 \geq \frac{R^2\sqrt{27(r^2-R^2)}}{2}$, we have $V=\frac{1}{3}\pi r^2\cdot\frac{rR}{\sqrt{r^2-R^2}}=\frac{1}{3}\pi r^3\cdot\frac{R}{\sqrt{r^2-R^2}}\geq\frac{1}{3}\pi\cdot\frac{R^2\sqrt{27(r^2-R^2)}}{2}\cdot\frac{R}{\sqrt{r^2-R^2}}=\frac{\sqrt{3}}{2}\pi R^3$. This completes the proof.