Mathematical Medley Problems Corner

A. Instruction

1. Prizes in the form of book vouchers will be awarded to one or more received best solutions submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

3. Solutions should be sent to :
   The Editor, Mathematics Medley,
   c/o Department of Mathematics,
   National University of Singapore,
   2 Science Drive 2, Singapore117543 ;
   and should arrive before 1 December 2002.

4. The Editor's decision will be final and no correspondence will be entertained.

B. Problems

Problem 1

Show that every positive integer can be written as

$$a_1 1^3 + a_2 2^3 + \cdots + a_k k^3$$

for $a_i \in \{\pm 1, \pm 2\}$ and some positive integer k.

(One $150 book voucher)

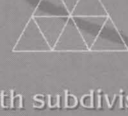
Problem 2

The nth subdivision of an equilateral triangle is the configuration obtained by

(i) dividing each side of the triangle into n equal parts by (n-1) points and

(ii) adding 3(n-1) line segments to join the 3(n-1) pairs of points in (i) on adjacent sides so that the line segments are parallel to the third side.

For example,

n = 2  n = 3 

How many triangles can you find in the 10th subdivision of an equilateral triangle?

(One $150 book voucher)

SOLUTIONS

C. Solutions to the problems of volume 28, No.2, 2001

Problem 1.

(Proposed by Dr Roger Poh, NUS) Prove that every positive rational number can be expressed as a finite series in the form of

$$\frac{1}{p_1}+\frac{1}{p_1p_2}+\frac{1}{p_1p_2p_3}+\ldots+\frac{1}{p_1p_2\cdots p_k}$$

where k, p_1, p_2, ..., p_k are positive integers with $p_1 \le p_2 \le \ldots \le p_k$.

(One $150 book voucher)

Solution

by Tan Weiyu, Colin - Raffles Junior College

Solved also by Joel Tay Wei En (Anglo Chinese School Independent) and Calvin Lin Zhiwei (Hwa Chong Junior College). The prize went to Colin Tan Weiyu.

Let s_1 be a given positive rational number. We can assume without loss of generality that $0 < s_1 < 1$, because if say $s_1 = n + s_1'$ where n is a positive integer and $0 \le s_1' < 1$, then $s_1 = \frac{1}{1}+\frac{1}{1.1}+\frac{1}{1.1.1.}+\cdots+\frac{1}{1^n}+s_1'$. If $s_1' = 0$, then we are done; if $s_1' > 0$, then s_1' is expressible as a finite series satisfying the given conditions implies s_1 is expressible in the same manner too.

Write $s_1 = \frac{a_1}{b_1}$ where a_1 and b_1 are coprime positive integers with $a_1 < b_1$. By a variant of the division algorithm, there exists unique positive integers α and β satisfying $b_1 = \alpha a_1 + \beta, 0 < \beta \le a_1$. Then set $p_1 = \alpha + 1 > 1$, so we have $(p_1 - 1)a_1 < b_1 \le p_1 a_1$. Thus $a_1 p_1 - b_1 < a_1$ and $a_1 p_1 - b_1 \ge 0$.

Let $s_2 = p_1\left(s_1 - \frac{1}{p_1}\right) = p_1\left(\frac{a_1}{b_1} - \frac{1}{p_1}\right) = \frac{a_1 p_1 - b_1}{b_1} \ge 0$. If $s_2 = 0$ then we are done, as we have $s_1 = \frac{1}{p_1}$. Otherwise, $s_2 > 0$, and writing $s_2 = \frac{a_2}{b_2}$ where a_2 and b_2 are coprime positive integers with $a_2 < b_2$, we can find a unique positive integer $p_2 > 1$ such that $a_2 p_2 - b_2 < a_2$ and $a_2 p_2 - b_2 \ge 0$, as before. But $s_2 = \frac{a_2}{b_2} = \frac{a_1 p_1 - b_1}{b_1}$ implies $b_1 = \frac{b_2(a_1 p_1 - b_1)}{a_2} \in \mathbb{N}$, so $a_2 \mid b_2(a_1 p_1 - b_1)$, thus $a_2 \mid (a_1 p_1 - b_1)$ since a_2 and b_2 are coprime. From this, we conclude that $a_2 \le a_1 p_1 - b_1 < a_1$, so the numerator of s_2 is strictly smaller than the numerator of s_1. Also note that $s_2 = \frac{a_1 p_1 - b_1}{b_1} < \frac{a_1}{b_1} = s_1$, then $p_2 \ge p_1$, otherwise $p_2 \le p_1 - 1$, so $s_2 = \frac{a_2}{b_2} = \frac{a_2 p_2}{b_2 p_2} \ge \frac{b_2}{b_2 p_2} = \frac{1}{p_2} \ge \frac{1}{p_1 - 1} = \frac{a_1}{a_1(p_1 - 1)} > \frac{a_1}{b_1} = s_1$, a clear contradiction.

Again, let $s_3 = p_2\left(s_2 - \frac{1}{p_2}\right) \ge 0$. If $s_3 = 0$, then we are done as $s_1 = \frac{1}{p_1} + \frac{s_2}{p_1} = \frac{1}{p_1} + \frac{1}{p_1 p_2}$. Otherwise if $s_3 > 0$, we continue the process, getting a sequence $s_1 = \frac{a_1}{b_1}$, $s_2 = \frac{a_2}{b_2}$, $s_3 = \frac{a_3}{b_3}$, ... where $a_1 > a_2 > a_3 > \cdots$. But this sequence is finite and since all the a_i's are positive integers, there must exist a positive integer k such that $a_k = 1$. This implies $s_{k+1} = 0$, so the process ends and we have

$$s_1 = \frac{1}{p_1} + \frac{1}{p_1 p_2} + \frac{1}{p_1 p_2 p_3} + \ldots + \frac{1}{p_1 p_2 \cdots p_k}.$$

<u>Editor's note:</u> This problem can also be solved elegantly by induction:

Let m be any given positive integer. For every positive integer n, let $P(n)$ denote the statement that $\frac{n}{m}$ can be expressed as a finite series in the required form with $p_1 = \left\lceil \frac{m}{n} \right\rceil$. It suffices to prove that $P(n)$ is true for all $n \in \mathbb{N}$ by induction on n. For $n = 1$, $P(1)$ is clearly true. Assume that $P(n)$ is true for $1 \le n \le \ell$, where $\ell \ge 1$. For $n = \ell + 1$, we have $m = (\ell+1)q - r$, where $q = \left\lceil \frac{m}{\ell+1} \right\rceil$ and $0 \le r \le \ell$. If $r = 0$, then $\frac{\ell+1}{m} = \frac{1}{q}$ and we are done. If $1 \le r \le \ell$, then by induction hypothesis, $P(r)$ is true. Hence,

$$\begin{aligned}\frac{\ell+1}{m} &= \frac{1}{q} + \frac{1}{q}\left(\frac{r}{m}\right)\\ &= \frac{1}{q} + \frac{1}{q}\left(\frac{1}{q_1} + \frac{1}{q_1 q_2} + \cdots + \frac{1}{q_1 q_2 \cdots q_t}\right)\\ &= \frac{1}{p_1} + \frac{1}{p_1 p_2} + \frac{1}{p_1 p_2 p_3} + \cdots + \frac{1}{p_1 p_2 \cdots p_k}\end{aligned}$$

where $k = t + 1 \in \mathbb{N}$, $p_1 = q, p_2 = q_1, p_3 = q_2, \ldots, p_k = q_t$. Also, $\frac{m}{\ell+1} \le \frac{m}{r}$ implies that $p_1 = q = \left\lceil \frac{m}{\ell+1} \right\rceil \le \left\lceil \frac{m}{r} \right\rceil = q_1 = p_2$. Thus, $p_1 \le p_2 \le \cdots \le p_k$ and the proof is complete.

SOLUTIONS

C. Solutions to the problems of volume 28, No.2, 2001

Problem 2.

Find all positive prime integers n such that for each positive prime integer $p < n$, $n - \left\lfloor \frac{n}{p} \right\rfloor p$ contains no square factors.

(One \$150 book voucher)

Solution

by Calvin Lin Zhiwei – Hwa Chong Junior College

We can easily check that $n = 2, 3, 5$ and 7 are solutions. For larger values of n, consider $n-4$.

If $n-4$ contains a prime p larger than 3, then $n - \left\lfloor \frac{n}{p} \right\rfloor p = 4$, which is a square factor.

Hence, the prime factorization of $n-4$ does not contain any prime larger than 3. As $n-4$ is odd, $n-4=3^k$ for some integer value k.

Consider $n-9=3^k-5$. If $n-9$ contains a prime larger than 7, $n - \left\lfloor \frac{n}{p} \right\rfloor p = 9$, which is a square factor. Hence, the prime factorization of $n-9$ contains only 2, 3, 5 and 7. Since 3^k-5 is clearly not divisible by 3 nor 5, $n-9=3^k-5=2^x 7^y$ for some integer values x and y.

Considering the equation modulo 8, $3^k-5 \equiv 3^k+3 \equiv 6$ or $4 \pmod 8$. Thus, $x=1$ or 2. If $x=1$, considering modulo 3, $3^k-5=2\times 7^y \Leftrightarrow 1 \equiv (-1)(1)^y \pmod 3$. Since the congruence on the right side would never be true, $x=1$ leads to no solutions. If $x=2$, considering modulo 4, $3^k-5=4\times 7^y \Leftrightarrow 1(-1)^k-1 \equiv 0 \pmod 4$ implies that k is even. Set $k=2j$, then by considering modulo 7, if $y \ge 1$, $9^j-5=4\times 7^y \Leftrightarrow 2^j \equiv 5 \pmod 7$. Since $2^a \equiv 2, 4$ or $1 \pmod 7$, there are no solutions if $y \ge 1$. If $x=2$, $y=0$, then $n=13$. A quick check shows that this is also a solution. Thus, the only solutions are $n = 2, 3, 5, 7$ and 13.

Editor's note: Solved also by Meng Dazhe (Raffles Junior College). The prize went to Calvin Lin Zhiwei.