Do we need Mean Value Theorem to prove $f'(x) = 0$ on (a,b) implies that f = constant on (a,b)?

Ng Tze Beng

Department of Mathematics
National University of Singapore

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If we use the *Mean Value Theorem* here, then it is an immediate consequence of it. What does that mean? Basically that means the *Mean Value Theorem* does all the work for us. So how is the *Mean Value Theorem* proved? One proof involves the use of the *Extreme Value Theorem.* How is that proved? It involves the use of the completeness property of the real numbers. So we can ask the question: If we can define the notion of differentiability for a function from a non complete ordered field such as the rational numbers into itself, then does the *Mean Value Theorem* hold? We can obviously find examples of function from the rational numbers to the rational numbers where the *Mean Value Theorem* or *Rolle's Theorem* does not hold. An easy example would be a cubic polynomial function whose derived function is a quadratic with real non-rational roots, for instance $f(x)=x^3-6x+1$. Is there a function from the rational numbers or an appropriate subset of it to the rational numbers whose derived function is zero but *f* is non-constant? An appropriate subset would be an intersection of a non-empty open interval with the rational numbers. Think of the holes that the rational numbers have. An easy example would be a function *f* defined by $f(x)=1$ for any rational number $x>\sqrt{2}$ and $f(x)=2$ for any rational number $x<\sqrt{2}$. *f* is not a constant function. Then the function $f:\mathbf{Q}\to\mathbf{Q}$ is differentiable and $f'(x)=0$ for any rational number *x*. A more sophisticated example will be provided by $g:(-\sqrt{2},\sqrt{2})\cap\mathbf{Q}\to\mathbf{Q}$ where $g(x)=1/2^{2n+2}$ for $x\in(\sqrt{2}/2^{n+1},\sqrt{2}/2^{n})\cap\mathbf{Q}$, or $x\in(-\sqrt{2}/2^{n},-\sqrt{2}/2^{n+1})\cap\mathbf{Q}$, *n* an integer ≥ 0 and $g(0)=0$. Then *g* is differentiable and $g'(x)=0$ for all *x* in $(-\sqrt{2},\sqrt{2})\cap\mathbf{Q}$ and *g* is not a constant function.

Theorem 1. $f'(x)=0$ on (a,b) implies that $f=$ constant on (a,b).

Now we prove the above using only the completeness property of the real numbers. We assume $b>a$. The proof is by contradiction. Suppose that *f* is not constant. Then there exist *u*, *v* in (a,b), $u<v$ such that $f(u)\neq f(v)$. This means $f(v)-f(u)\neq 0$. Then we shall make use of the difference quotient $\dfrac{f(v)-f(u)}{v-u}=C\neq 0$ to deduce a contradiction. Suppose now that $C>0$.

For now let us suppose that (not assuming anything on *C*, i.e. *C* can be any real number.)

$$f(v)-f(u)=C(v-u). \qquad (*)$$

We are going to bisect the interval $[u,v]$, pick the next interval from this bisection and continue bisecting in like manner.

Take the mid point $w = \dfrac{u+v}{2}$ of $[u,v]$. Then either

$$f(v) - f(w) \geq C(v-w) \tag{1}$$

or

$$f(w) - f(u) \geq C(w-u). \tag{2}$$

This is because if both (1) and (2) do not hold, then we would have

$$f(v) - f(w) < C(v-w) \quad \text{and} \quad f(w) - f(u) < C(w-u),$$

which would imply that $f(v) - f(u) < C(v-u)$ contradicting (*).

If (1) holds, then we name $u_1 = w$ and $v_1 = v$. If (1) does not hold we name $u_1 = u$ and $v_1 = w$.

Let $k = (v-u)$. Then $|v_1 - u_1| = k/2$ and

$$f(v_1) - f(u_1) \geq C(v_1 - u_1). \tag{*1}$$

Obviously, $[u_1, v_1] \subset [u, v]$, $u \leq u_1 < v_1 \leq v$, $|u_1 - u| \leq |v-u|/2 = k/2$ and $|v - v_1| \leq |v-u|/2 = k/2$. We next take the mid point $w_1 = \dfrac{u_1 + v_1}{2}$ of $[u_1, v_1]$. Then we shall have either

$$f(v_1) - f(w_1) \geq C(v_1 - w_1) \tag{3}$$

or

$$f(w_1) - f(u_1) \geq C(w_1 - u_1). \tag{4}$$

Again this is because if both (3) and (4) do not hold then we would have $f(v_1) - f(w_1) < C(v_1 - w_1)$ and $f(w_1) - f(u_1) < C(w_1 - u_1)$ implying $f(v_1) - f(u_1) < C(v_1 - u_1)$ thus contradicting (*1).

If (3) holds, then we name $u_2 = w_1$ and $v_2 = v_1$. If (3) does not hold we name $u_2 = u_1$ and $v_2 = w_1$. Then $|v_2 - u_2| = k/2^2$,

$$f(v_2) - f(u_2) \geq C(v_2 - u_2). \tag{*2}$$

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Obviously, $[u_2,v_2]\subset[u_1,v_1]$, $u_1\le u_2<v_2\le v_1$, $|u_2-u_1|\le|v_1-u_1|/2=k/2^2$ and $|v_1-v_2|\le|v_1-u_1|/2=k/2^2$.

In this way we obtained a nested sequence

$$\ldots\subset[u_n,v_n]\subset\ldots\ldots\subset[u_2,v_2]\subset[u_1,v_1]\subset[u,v]$$

with the length of the interval $[u_n,v_n]$, $\dfrac{v-u}{2^n}$ approaches 0 as n tends to infinity; an increasing sequence (not necessarily strictly increasing)

$$u_1\le u_2\le u_3\le\ldots\ldots\le u_n\le\ldots$$

satisfying, for all n, $u_n<v_n\le v$,

$$|u_n-u_{n-1}|\le k/2^n \qquad (5)$$

and a decreasing sequence (not necessarily strictly decreasing)

$$v_1\ge v_2\ge v_3\ge\ldots\ldots\ge v_n\ge\ldots$$

satisfying, for all n, $u\le u_n<v_n$,

$$|v_n-v_{n-1}|\le k/2^n \qquad (6)$$

and

$$f(v_n)-f(u_n)\ge C(v_n-u_n). \qquad (*n)$$

Now we have a choice to proceed. We can use the Weierstrass characterization of completeness to conclude that the nested sequence $\{[u_n,v_n]\}_n$ must have a unique intersection i.e, there is exactly one point x that belongs to $[u_n,v_n]$ for all n. (See [2]. For a list of equivalence of the completeness property see [1]. For a less demanding reference see [3].) We can also note that the sequence or set $\{u_n\}$ is bounded above by v by (5). Therefore, by the completeness property of the real numbers, $\{u_n\}$ has a least upper bounded or supremum in **R** also denoted by x, i.e. $x=\sup\{u_n\}$. Also by the completeness property of the real numbers since the sequence $\{v_n\}$ is bound below by u by (6) it has a greatest lower bound or infimum in **R** denoted by y, that is, $y=\inf\{v_n\}$.

We claim that $x = y$. From (5) any v_n is an upper bound for $\{u_n\}$. Hence $x = \sup\{u_n\} \le v_n$ for each n. Therefore, x is a lower bound for $\{v_n\}$ and so $x \le y = \inf\{v_n\}$. Can x be bigger than y? Suppose $x > y$. Then since $x = \sup\{u_n\}$, there exists a u_j such that $y < u_j$. But since $y = \inf\{v_n\}$ and $u_j < v_n$ for all n, $u_j \le y = \inf\{v_n\}$. This contradicts $y < u_j$. Hence $x = y$. In particular, we have $u_n \le x \le v_n$ for all n. That is the same as saying $x \in [u_n, v_n]$ for all n.

Next we shall show that $f'(x) \ge C$. That is $\lim_{y\to x} \frac{f(y)-f(x)}{y-x} \ge C$. If on the contrary $\lim_{y\to x} \frac{f(y)-f(x)}{y-x} < C$, then there exists a $\delta > 0$ such that for all y with $0 < |y-x| < \delta$ we have

$$\frac{f(y)-f(x)}{y-x} < C. \qquad (A)$$

If we can show that for any $\delta > 0$, we can find a x_δ such that $0 < |x_\delta - x| < \delta$ but $\frac{f(x_\delta)-f(x)}{x_\delta - x} \ge C$. Then no $\delta > 0$ can exist so that (A) holds and so we can conclude that $f'(x) \ge C$. We shall now proceed to do just that.

For any $\delta > 0$, $x - \delta < x = \sup\{u_n\}$ and so there exists integer N such that $x - \delta < u_N \le x$. Likewise using the fact that $x = \inf\{v_n\}$, there exists an integer M such that $x \le v_M < x + \delta$. Let $K = \max(N, M)$. Then we have

$$x - \delta < u_N \le u_K \le x \le v_K \le v_M < x + \delta$$

and

$$u_K < v_K.$$

This means that both u_K and v_K lie in the interval $(x-\delta, x+\delta)$. If $x = u_K$, then let $x_\delta = v_k$. If $x = v_K$, then let $x_\delta = u_K$. In either case using $(*_K)$, we have

$$\frac{f(x_\delta)-f(x)}{x_\delta - x} = \frac{f(v_K)-f(u_K)}{v_K - u_K} \ge C.$$

If $u_K < x < v_K$, then as in the beginning of the proof one of the following must be true:

$$f(v_K) - f(x) \ge C(v_K - x) \qquad (7)$$

$$f(x) - f(u_K) \ge C(x - u_K). \qquad (8)$$

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This is because if both (7) and (8) do not hold, we would then get

$$f(v_K) - f(x) < C(v_K - x)$$

and $f(x) - f(u_K) < C(x - u_K)$ implying that $f(v_K) - f(u_K) < C(v_K - u_K)$ *contradicting* (*K*). If (7) holds, then we let $x_\delta = v_k$ and if (8) holds we let $x_\delta = u_K$. We then have

$$\frac{f(x_\delta) - f(x)}{x_\delta - x} \geq C. \quad (9)$$

Hence we conclude that if $C > 0$ this would give us $f'(x) \geq C > 0$ thus contradicting $f'(x) = 0$. Thus $C \leq 0$.

Suppose $C < 0$. We can either apply the above argument with the inequality "≥" replaced by "≤" throughout or we can consider using the function $g = -f$. We can rewrite (*) as

$$-f(v) - (-f(u)) = -C(v - u).$$

That is

$$g(v) - g(u) = (-C)(v - u). \quad (**)$$

Now $-C > 0$ and so (**) is similar to (*) and so we can conclude that we can find an x in $[u, v] \subseteq (a, b)$ such that $g'(x) = -f'(x) \geq -C$, that is $f'(x) \leq C < 0$ thus contradicting $f'(x) = 0$. Therefore, $C = 0$ and so f must be a constant function.

Note that we have actually proved the following result:

Theorem 2. If $f : [a, b] \to \mathbf{R}$ is differentiable, then for any u, v in $[a, b]$ with $u < v$ there exists a point x in $[u, v]$ such that $f'(x) \geq \dfrac{f(v) - f(u)}{v - u}$.

Reversing the inequality "≥" by "≤" throughout, starting with (1) and (2) we would obtain the following:

Theorem 2′. If $f : [a, b] \to \mathbf{R}$ is differentiable, for any u, v in $[a, b]$ with $u < v$ there exists a point x in $[u, v]$ such that $f'(x) \leq \dfrac{f(v) - f(u)}{v - u}$.

Theorem 3. If $f'(x) < 0$ on (a, b), then f is decreasing on (a, b).

Proof. Take any u, v in (a, b) with $u < v$, then by Theorem 2, there exists a point x in the interval $[u, v]$ such that $\frac{f(v)-f(u)}{v-u} \le f'(x) < 0$. Hence $f(v)-f(u)<0$ and so $f(v)<f(u)$. That means f is decreasing on (a, b).

Theorem 4 (Weak Mean Value Theorem). If $m \le f'(x) \le M$ on $[a, b]$, then for any u, v in $[a, b]$ with $u < v$,

$$m(v-u) \le f(v)-f(u) \le M(v-u).$$

Proof. By Theorem 2, $f(v)-f(u) \le f'(y)(v-u)$ for some y in $[u, v]$ and so $f(v)-f(u) \le M(v-u)$. By Theorem 2′, there is a point y in $[u, v]$ such that $f(v)-f(u) \ge f'(y)(v-u) \ge m(v-u)$. Therefore, $m(v-u) \le f(v)-f(u) \le M(v-u)$.

References

[1] B. Artmann, *The Concept of Numbers*, Ellis Horwood Limited, (Chapter 1E).

[2] Ebbinghaus et al., *Numbers*, Springer (Chapter 2, Section 5).

[3] Patrick M. Fitzpatrick, *Advanced Calculus*. PWS Publishing Company, (Chapter 1).

Ng Tze Beng

Ng Tze Beng is an associate professor at the Mathematics Department of NUS, author of the text book, Calculus, an Introduction published by Springer Verlag, maintains a comprehensive calculus web site at http:www.math.nus.edu.sg/~matngtb/Calculus, keenly interested in calculus and Computer Algebra, Lab based instructions, IT in learning and teaching, particularly in the use of software, Derive for Windows in teaching, demonstrating, modeling mathematics and application of mathematics. Research interests: vector fields, H-spaces, cohomology operations.