COMPETITION CORNER

In this issue we publish the problems of the Canadian Mathematical Olympiad 1993 and the Singapore IMO National Team Selection Test 2002.

Please send your solutions of these Olympaids to me at the address given. All correct solutions will be acknowledged. We also present solutions of the Mathematical Competitions in Croatia 2000, Bulgarian Mathematical Olympaid 1994, and the 42nd International Mathematical Olympiad.

All communications about this column should be sent to :

Dr. Tay Tiong Seng,
Department of Mathematics,
National University of Singapore,
2 Science Drive 2,
Singapore 117543
or by e-mail to mattayts@nus.edu.sg.

• Problems •

Canadian Mathematical Olympiad 1993

1. Determine a triangle whose three sides and an altitude are four consecutive integers and for which this altitude partitions the triangle into two right triangles with integer sides. Show that there is only one such triangle.

2. Show that the number x is rational if and only if three distinct terms that form a geometric progression can be chosen from the sequence

$$x,\ x+1,\ x+2,\ x+3,\cdots.$$

3. In triangle ABC, the medians to the sides AB and AC are perpendicular. Prove that $\cot B + \cot C \geq \frac{2}{3}$.

4. A number of schools took part in a tennis tournament. No two players from the same school played against each other. Every two players from different schools played exactly one match against each other. A match between two boys or between two girls was called a *single* and that between a boy and a girl was called a *mixed single*. The total number boys differed from the total number of girls by at most 1. The total number of singles differed from the total number of mixed singles by at most 1. At most how many schools were represented by an odd number of players?

5. Let $y_1, y_2, y_3, \ldots$ be a sequence such that $y_1 = 1$ and, for $k > 0$, is defined by the relationship:

$$y_{2k} = \begin{cases} 2y_k & \text{if } k \text{ is even} \\ 2y_k + 1 & \text{if } k \text{ is odd} \end{cases}$$

$$y_{2k+1} = \begin{cases} 2y_k & \text{if } k \text{ is odd} \\ 2y_k + 1 & \text{if } k \text{ is even.} \end{cases}$$

Show that the sequence $y_1, y_2, y_3, \ldots$ takes on every positive integer value exactly once.

Singapore IMO National Team Selection Tests 2002

Note: Problems 1 to 6 were used to select the national team trainees. The last 11 problems were used to select the final team of 6 to represent Singapore at the International Mathematical Olympiad to be held at Glasgow, United Kingdom in July 2002.

1. Let A, B, C, D, E be five distinct points on a circle Γ in the clockwise order and let the extensions of CD and AE meet at a point Y outside Γ. Suppose X is a point on the extension of AC such that XB is tangent to Γ at B. Prove that $XY = XB$ if and only if XY is parallel DE.

2. Let n be a positive integer and $(x_1, x_2, \ldots, x_{2n})$, $x_i = \pm 1$, $i = 1, 2, \ldots, 2n$ be a sequence of $2n$ integers. Let S_n be the sum

$$S_n = x_1 x_2 + x_3 x_4 + \cdots + x_{2n-1} x_{2n}.$$

If O_n is the number of sequences such that S_n is odd and E_n is the number of sequences such that S_n is even, prove that

$$\frac{O_n}{E_n} = \frac{2^n - 1}{2^n + 1}.$$

3. For every positive integer n, show that there is a positive integer k such that

$$2k^2 + 2001k + 3 \equiv 0 \pmod{2^n}.$$

4. Let x_1, x_2, x_3 be positive real numbers. Prove that

$$\frac{(x_1^2 + x_2^2 + x_3^2)^3}{(x_1^3 + x_2^3 + x_3^3)^2} \leq 3.$$

5. For each real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x. For example $\lfloor 2.8 \rfloor = 2$. Let $r \geq 0$ be a real number such that for all integers m, n, $m|n$ implies $\lfloor mr \rfloor | \lfloor nr \rfloor$. Prove that r is an integer.

6. Find all functions $f : [0, \infty) \longrightarrow [0, \infty)$ such that $f(f(x)) + f(x) = 12x$, for all $x \geq 0$.

7. Let P a point on the extension of the diagonal AC of rectangle $ABCD$ beyond C, such that $\angle BPD = \angle CBP$. Determine the value of $\frac{PB}{PC}$.

8. Let $a_1, a_2 \ldots, a_n$ be positive real numbers and let $A = \sum a_i$. Prove that

$$\sum \frac{a_i}{2A - a_i} \geq \frac{n}{2n - 1}.$$

9. Suppose the sum of m pairwise distinct positive even numbers and n pairwise distinct positive odd numbers is 2002. What is the maximum value of $3m + 4n$?

10. Determine the number of positive integers that can be expressed in the form
$$\frac{1}{a_1}+\frac{2}{a_2}+\cdots+\frac{2002}{a_{2002}},$$
where $a_1, a_2 \ldots, a_{2002}$ are positive integers.

11. Prove the inequality
$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} b_i\right) \geq \left(\sum_{i=1}^{n}(a_i+b_i)\right)\left(\sum_{i=1}^{n}\frac{a_i b_i}{a_i+b_i}\right)$$
for any positive numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$.

12. Two circles Γ_1 and Γ_2 are tangent to each other internally at a point N such that Γ_2 is inside Γ_1. Points C, S and T are on Γ_1 such that CS and CT are tangent to Γ_2 at M and K respectively. Let U and V be the midpoints of the arcs CS and CT respectively. Prove that $UCVW$ is a parallelogram where W is the second point of intersection between the circumcircles of $\triangle UMC$ and $\triangle VCK$.

13. In triangle ABC, let P and Q lie on the interior of BC such that $\angle BAP = \angle CAQ$. Let I be the incentre of ABC. Also, let J and K be the incentres of triangles BAP and CAQ respectively. Prove that AI, BK and CJ are concurrent.

14. Prove that for any $n \geq 2$ distinct positive integers $a_1, a_2, \ldots, a_n$,
$$\prod_{1\leq j<i\leq n}\frac{a_i-a_j}{i-j}$$
is an integer.

15. A set of three nonnegative integers $\{x, y, z\}$ with $x < y < z$ is called *historic* if $\{z-y, y-x\} = \{1819, 1965\}$. Show that the set of all nonnegative integers can be written as the union of pairwise disjoint historic sets.

16. A set S of nonnegative real numbers is said to be *good* if for any $x, y \in S$, either $x+y \in S$ or $|x-y| \in S$. For example, if r is a positive real number and n is a positive integer, the set $S(n,r) = \{0, r, 2r, \ldots, nr\}$ is *good*. Prove that any finite *good* set which is not the set $\{0\}$ is either of the form $S(n,r)$ or has exactly 4 elements.

17. Find all positive integers n such that $2^n - 1$ is a multiple of 3 and $(2^n-1)/3$ is a divisor of $4m^2+1$ for some integer m.

• Solutions •

Mathematical Competitions in Croatia 2000

Selected problems

1. Find all positive integer solutions of the equation

$$\frac{1}{x}+\frac{2}{y}-\frac{3}{z}=1.$$

Solution Calvin Lin Ziwei (Hwachong Junior College).

Note that $\frac{1}{x}+\frac{2}{y}=1+\frac{3}{z}>1$. When $x=1$, we get $2z=3y$ and so $(x,y,z)=(1,2k,3k)$ is a solution for $k\in\mathbb{N}$.

When $x\geq 2$, we have $\frac{1}{2}+\frac{2}{y}\geq\frac{1}{x}+\frac{2}{y}>1$ and whence, $y<4$. By considering the cases $y=1,3,2$, we get $(x,y,z)=(2,1,2),(2,3,18),(k,2,3k)$, $k\in\mathbb{N}$ as solutions.

2. The incircle of $\triangle ABC$ touches its sides BC, CA, and AB in the points A_1, B_1 and C_1, respectively. Determine the angles of $\triangle A_1B_1C_1$ in terms of angles of $\triangle ABC$.

Wee Hoe Teck (Singapore) and Colin Tan Weiyu (Raffles Junior College) both have similar solutions.

Note that $\angle AC_1B_1=\angle C_1B_1A=90^\circ-A/2$, since $\triangle AB_1C_1$ is isosceles. Therefore, the angles of $\triangle A_1B_1C_1$ are $90^\circ-A/2, 90^\circ-B/2, 90^\circ-C/2$ respectively.

3. Let $ABCD$ be a square with side length 20. Let T_i, $i=1,2,\ldots,2000$, be points in its interior so that no three points from the set $S=\{A,B,C,D\}\cup\{T_i : i=1,2,\ldots,2000\}$ are collinear. Prove that at least one triangle with vertices in S has area less than $\frac{1}{10}$.

The following was submitted jointly by Wee Hoe Teck and Thevandran Senkodan (Singapore).

Lemma: Given any set S of $n\geq 3$ points on the plane, no 3 collinear, there exists a triangulation of the convex hull of S.

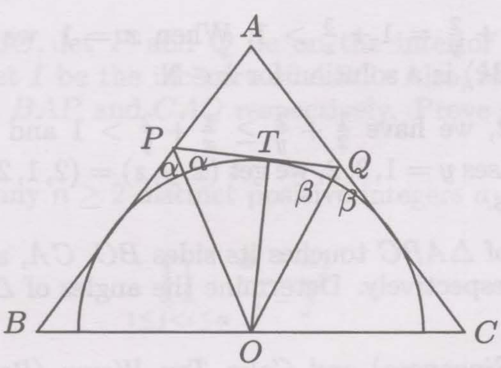
Proof of Lemma: The proof is by induction. The case $n=3$ is trivial. Assume the result holds for up to n points. Consider S comprising $n+1$ points. If all $n+1$ points lie on the boundary of the convex hull, then fixing any one point and joining the edges between that point and every other point on the convex hull yields a triangulation. If the convex hull contains $\leq n$ points, pick some point in the interior of the convex hull and join it to each point on the convex hull. Apply the induction hypothesis to triangulate each triangle that's formed. The result follows.

Main Solution: Consider a triangulation of the square $ABCD$ by the points T_i, $i = 1, 2, \ldots, 2000$. Applying Euler's Formula $V - E + F = 1$, where $V = 2004$ and $3F = 2E - 4$ by a counting argument (in $3F$, every interior edge is counted exactly twice, the 4 edges of $ABCD$ are counted exactly once each), we have $F = 4002$. Therefore, by Pigeonhole Principle, there exists some triangle whose area is at most $20 \times 20/4002 < 1/10$.

4. The circle with centre on the base BC of an isosceles triangle ABC is tangent to equal sides AB, and AC. Let P and Q be points on the sides AB and AC, respectively. Prove that

$$PB \cdot CQ = \frac{BC^2}{4}$$

if and only if PQ is tangent to this circle.



Solutions were received from Calvin Lin Ziwei (Hwachong Junior College), whose solution is presented below, and David Pargeter (England).

It's easy to see that the centre O of the circle is at the midpoint of the side BC. If PQ is tangent to the circle at T, then $\angle OPQ = \angle OPB = \alpha$, $\angle OQP = \angle OQC = \beta$. Thus $\angle ABC = \angle ACB = \frac{1}{2}(360° - 2\alpha - 2\beta) = 180° - \alpha - \beta$. Thus $\angle POB = \beta$ and $\angle QOC = \alpha$. Therefore $\triangle PBO \equiv \triangle OCQ$. Thus $PB/BO = CO/CQ$ or $PB \cdot CQ = BC^2/4$.

Now suppose $PB \cdot QC = \frac{1}{4}BC^2$, let the tangent from P to the circle meet AC at R. Then by the above demonstration, $PB \cdot RC = \frac{1}{4}BC^2$ and this implies that R and Q coincide, so PQ is a tangent.

5. Let $n (\geq 3)$ positive integers be written on a circle so that each of them divides the sum of its neighbours. Denote

$$S_n = \frac{a_n + a_2}{a_1} + \frac{a_1 + a_3}{a_2} + \ldots \frac{a_{n-2} + a_n}{a_{n-1}} + \frac{a_{n-1} + a_1}{a_n}.$$

Determine the maximum and minimum of S_n.

Wee Hoe Teck (Singapore) contributed the following solution. Colin Tan Weiyu (Raffles Junior College) also found the minimum with a similar argument.

For any positive numbers x, y, we have $\frac{x}{y} + \frac{y}{x} \geq 2$ by AM-GM inequality. Thus,

$$S_n = \sum_{i=1}^{n} \left(\frac{a_{i+1}}{a_i} + \frac{a_i}{a_{i+1}} \right) \geq 2n$$

writing $a_{n+1} = a_1$. Also, for $a_1 = a_2 = \ldots = a_n = 1$, we have $S_n = 2n$. Therefore, the minimum of S_n is $2n$.

Next, we shall prove by induction that the maximum of S_n is $3n - 1$. For $n = 3$, assume without loss of generality that $a_1 \leq a_2 \leq a_3$. Therefore, $a_3 \mid (a_1 + a_2) \leq 2a_3$ implies either $a_1 + a_2 = 2a_3$ or $a_1 + a_2 = a_3$. In the first case, we must have $a_1 = a_2 = a_3$, which yields $S_3 = 6$. In the second case, $a_2 \mid (a_1 + a_3) = 2a_1 + a_2$, so $a_1 \leq a_2 \mid 2a_1$, which yields either $a_2 = a_1$ and thus $S_3 = 7$, or $a_2 = 2a_1, a_3 = 3a_1$ and thus $S_3 = 8$. Hence, the maximum of S_3 is 8, which holds for instance when $a_1 = 1, a_2 = 3, a_3 = 3$.

Now suppose the maximum of S_n is $3n - 1$, and consider any $n + 1$ numbers $a_1, \ldots, a_{n+1}$ in a circle such that each of them divides the sum of its neighbours. Without loss of generality, assume $a_{n+1} = \max\{a_1, \ldots, a_{n+1}\}$. Again, we have $a_{n+1} \mid a_1 + a_n \leq 2a_{n+1}$, which implies either $a_1 + a_n = 2a_{n+1}$ or $a_1 + a_n = a_{n+1}$. In the first case, we must have $a_1 = a_n = a_{n+1}$. In addition, $a_1 \mid a_{n+1} + a_2$ and $a_1 < a_{n+1} + a_2 \leq 2a_1$. Therefore, $a_2 = a_1$. By an inductive argument, we have $a_1 = a_2 = \cdots = a_{n+1}$ and thus $S_n = 2n$ in this case.

In the second case, that is, $a_1 + a_n = a_{n+1}$, we have $a_1 \mid (a_2 + a_{n+1}) = a_1 + a_2 + a_n$, and thus $a_1 \mid (a_1 + a_n)$. Similarly, $a_n \mid (a_{n-1} + a_{n+1})$ yields $a_n \mid (a_{n-1} + a_1)$. Therefore, the numbers $a_1, \ldots, a_n$ if written a circle satisfy the property that each of them divides the sum of its neighbors. It follows from our induction hypothesis that $S_n \leq 3n - 1$. Furthermore, substituting $a_1 + a_n = a_{n+1}$ into our expression for S_{n+1} yields $S_{n+1} = 3 + S_n \leq 3(n+1) - 1$. To see that this maximum can be achieved, take any $a_1, \ldots, a_n$ that yields $S_n = 3n - 1$, and add $a_{n+1} = a_1 + a_n$. It is straight-forward to check that we still have the property that each of the $n+1$ numbers divides the sum of its neighbors, and that $S_{n+1} = 3(n + 1) - 1$.

We could therefore conclude that the maximum and minimum of S_n are $2n$ and $3n - 1$ respectively.

6. Let $S = \{k \in \mathbb{N} : a \in \mathbb{N}, a^2 \mid k \Rightarrow a = 1\}$. For any $n \in \mathbb{N}$, prove that

$$\sum_{k \in S} \lfloor \sqrt{n/k} \rfloor = n.$$

Note: For any real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

First we present the solution by Calvin Lin Ziwei (Hwachong Junior College)

Form a table whose columns are indexed by members of $S = \{s_1, s_2, \ldots\}$ and whose rows are indexed by the squares $\{1^2, 2^2, 3^2, \ldots\}$. The entry in (s_i, j^2) is $s_i j^2$. The following shows the initial parts of the table:

$$\begin{array}{c} \\ 1 \\ 4 \\ 9 \\ 16 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 5 & 6 & 7 & 10 \end{array} \\ \left(\begin{array}{ccccccc} 1 & 2 & 3 & 5 & 6 & 7 & 10 \\ 4 & 8 & 12 & 20 & 24 & 28 & 40 \\ 9 & 18 & 27 & 45 & 54 & \ldots & \\ 16 & 32 & 48 & \ldots & & & \end{array}\right) \end{array}$$

Since any number can be uniquely written as s^2t, where t is square free, we see that the entries in the table are pairwise distinct. Clearly, there are n entries in the table that are less than or equal to n.

Consider the column indexed by k. If there are j numbers in the column less than or equal to n, then $\lfloor\sqrt{\frac{n}{k}}\rfloor = j$. This is true since $kj^2 \le n < k(j+1)^2$. Let k run through all the values in set S, we see that $\sum_{k\in S}\lfloor\sqrt{\frac{n}{k}}\rfloor = n$.

Wee Hoe Teck (Singapore) and Colin Tan Weiyu (Raffles Junior College) solved it in another way. We present a sketch below.

First observe that for any $n \in \mathbb{N}$ and $k \in S, k \le n$,

$$0 \le \left\lfloor\sqrt{\frac{n+1}{k}}\right\rfloor - \left\lfloor\sqrt{\frac{n}{k}}\right\rfloor \le 1$$

with the second equality holding if and only if $n+1 = a^2k$ for some $a \in \mathbb{Z}$, $a > 1$. The proof is straightforward.

The main result is then obtained by induction. Assume that the result holds for n. We only need to note two cases to complete the proof: $n+1 \in S$ and $n+1 \notin S$.

Bulgarian Mathematical Olympiad, 1994

Selected problems from competitions of various levels.

1. Thirty-three natural numbers are given. The prime divisors of each of the numbers are among $2, 3, 5, 7, 11$. Prove that the product of two of the numbers is a perfect square.

We present similar solutions by Ernest Chong Kai Fong and Tan Kiat Chuan (both from Raffles Junior College.)

Let $n = 2^{k_1}3^{k_2}5^{k_3}7^{k_4}11^{k_5}$ be a natural number, k_i is a non-negative integer for $i = 1, 2, \ldots, 5$. Let $f(n) = (a_1, a_2, a_3, a_4, a_5)$, where $a_i = 0$ if k_i is even, $a_i = 1$ if k_i is odd. Note that there are $2^5 = 32$ distinct possibilities for $f(n)$. Thus by pigeonhole-principle, there exist 2 natural numbers in the 33 natural numbers given, say p and q, such that $f(p) = f(q)$. Thus $f(p+q) = (0,0,0,0,0)$, i.e., $p+q$ is a perfect square.

2. Let
$$f(x) = x^4 - 4x^3 + (3+m)x^2 - 12x + 12$$
where m is a real number.

(a) Find all integers m such that the equation $f(x) - f(1-x) + 4x^3 = 0$ has at least one integer solution.

(b) Find all values of m such that $f(x) \geq 0$ for all real number x.

We present similar solutions by Tan Kiat Chuan (Raffles Junior College)and Calvin Lin Ziwei (Hwachong Junior College).

(a) From $f(x) - f(1-x) + 4x^3 = 0$ we get $6x^2 + (2m-26)x + (12-m) = 0$. Thus
$$x = \frac{13 - m \pm \sqrt{m^2 - 20m + 97}}{6}$$
whence $m^2 - 20m + 97$ must be perfect square. But if $m^2 - 20m + 97 = (m-10)^2 - 3 = k^2$ for some integer k, then $k = 1$ and $|m - 10| = 2$, i.e., $m = 8, 12$. When $m = 8$, $x = 1$ is an integer solution and when $m = 12$, $x = 0$ is an integer solution. Thus $m = 8, 12$.

(b) We have
$$f(x) = (x^2+3)(x-2)^2 + (m-4)x^2.$$
$f(x) \geq 0$ for all x if $m \geq 4$. If $m < 2$, then $f(2) = 16(m-4) < 0$. Therefore the answer is $m \geq 4$.

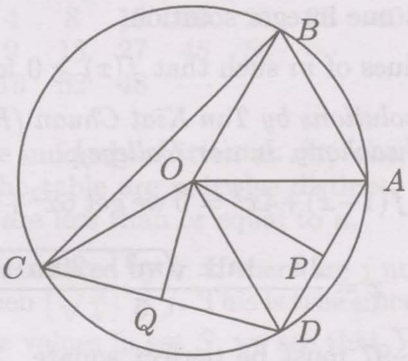
3. Let $\mathbb{N}_0$ be the set of nonnegative integers and $f(n)$ is a function $f : \mathbb{N}_0 \to \mathbb{N}_0$ such that $f(f(n)) + f(n) = 2n + 3$ for every $n \in \mathbb{N}_0$. Evaluate $f(1993)$.

We present a solution by induction. It's combined from similar solutions by Wee Hoe Teck (Singapore), Colin Tan Weiyu (Raffles Junior College) and Joel Tay Wei En (Anglo-Chinese School (Independent)). Tan Kiat Chuan (Raffles Junior College) solved the problem by converting it into a recurrence relation.

By putting $n = 0$,we get
$$f(0) + f^2(0) = 3.$$
Thus
$$(f(0), f^2(0) \in \{(0,3), (3,0), (1,2), (2,1)\}.$$
Clearly $f(0) \neq 0$. By putting $n = 3$, we find that $(3,0)$ is impossible. if $(f(0), f^2(0)) = (2,1)$, then $f(2) = 1$ and $f(1) = 6$. Then $f(1) + f^2(1) = 5$ implies $f^2(1) = -1$ which is impossible. Thus $f(0) = 1$. By putting $n = 0$, we get $f(1) = 2$. Now assume $f(n-1) = n$, $f(n) = n+1$ where $n \in \mathbb{N}$. Replacing n by $n-1$ in the given equation, we get $f(n+1) = n+2$. Thus it follows by induction that $f(n) = n+1$ for $n \geq 0$. Obviously then $f(1993) = 1994$.

4. A convex quadrilateral $ABCD$ is inscribed in a circle with centre O and diameter 25. P and Q are points on AD and CD, respectively, such that $OP \perp AD$ and $OQ \perp CD$. Find the lengths of the sides of $ABCD$ if the lengths of AB, BC, CD, DA, OP, OQ are distinct natural numbers.

We present the solution by Tan Kiat Chuan (Raffles Junior College). Wee Hoe Teck (Singapore) also obtained the answer.



Without loss of generality, we may let $AB \leq BC$ and $AD \leq CD$. By Pythagoras' Theorem, we have

$$CD^2 + (2OQ)^2 = 25^2.$$

But $x^2+(2y)^2 = 25^2$ has only two solutions: $(x, y) = (7, 12), (15, 10)$. Thus we have $AD = 7$, $OP = 12$, $CD = 15$, $OQ = 10$. Let $\angle ODA = \alpha$, $\angle ODC = \beta$, $\angle ABC = \theta$ $AB = x$ and $BC = y$. Then $\sin\theta = \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha + \sin\beta = 4/5$. Thus $\cos\theta = 3/5$ and $\cos\angle ADC = \cos(180° - \theta) = -3/5$. Apply the cosine rule to $\triangle ABC$ and $\triangle ADC$, we get

$$x^2 + y^2 - 2xy\cos\theta = AC^2 = AD^2 + CD^2 - 2(AD)(CD)\cos\angle ADC$$

and consequently

$$5x^2 - 6xy + 5y^2 = 2000.$$

Thus $x = \frac{3y+\sqrt{25^2-y^2}}{5}$. Since $25^2 - y^2$ is a perfect square when $y = 7, 15, 24, 25$. there are only two solutions: $(x, y) = (15, 25), (20, 24)$. The first is rejected. Thus the answer is $AB = 20$, $BC = 24$.

5. A point D lies on the side AB of $\triangle ABC$. The excircle k_1 of $\triangle ACD$, which touches the side CD externally, touches the sides AC and AD at points P and L, respectively. The excircle k_2 of $\triangle BCD$, which touches the side CD externally, touches the sides BC and BD at points Q and K, respectively. The incircle k_3 of $\triangle ACD$ touches the sides AC and AD at the points M and E, respectively and the incircle k_4 of $\triangle BCD$ touches the sides BC and BD at the points N and F, respectively.

(a) Prove that $FK = EL = MP = NQ$.

(b) If $\angle ACB = 90°$ determine the position of the point D so that the area of the convex quadrilateral $MNPQ$ is minimal.

Tan Kiat Chuan (Raffles Junior College) and Calvin Lin Ziwei (Hwachong Junior College) submitted similar solutions.

(a) Let $BC = a$, $AC = b$, $CD = m$, $AD = y$ and $DB = x$. By the properties of excircles and incircles, we have

$$2BQ = 2BK = a + x + m$$
$$2AP = 2AL = b + y + m$$
$$2BN = 2BF = a + x - m$$
$$2AM = 2AE = b + y - m$$

From here it is easy to obtain $FK = AL = MP = NQ = m$.

(b) If $\angle ACB = 90°$

$$\begin{aligned}\text{Area } MNPQ &= \frac{1}{2}[(MC)(NQ) + (CP)(NQ)] = \frac{1}{2}(NQ)[MC + CP] \\ &= \frac{1}{2}(NQ)(MP) = \frac{m^2}{2}\end{aligned}$$

Thus the area is minimal when m is minimal, i.e., when D is the foot of the altitude from C onto AB.

6. Let $n > 1$ be a natural number and

$$A_n = \{x \in \mathbb{N} : \gcd(x, n) \neq 1\}.$$

The number n is called *interesting* if for any $x, y \in A_n$, we have $x + y \in A_n$. Find all interesting n.

Similar solutions were received from Meng Dazhe, Tan Kiat Chuan (both from Raffles Junior College), Calvin Lin Ziwei (Hwachong Junior College) and Wee Hoe Teck (Singapore). Ernest Chong Kai Fong (Raffles Junior College) also has a solution along similar lines. We present Meng's solution.

The answer is: all interesting ns are powers of primes. When n is a power of prime, say p^i, any number not coprime to it must be a multiple of p also, say $x = px'$, $y = py'$. Then $x + y = p(x' + y')$, also a multiple of prime thus not coprime to n.

Suppose there exist an interesting n which is not a power of prime. Then n has two distinct prime divisors p and q. Let $n = p_1^{i_1} p_2^{i_2} \dots p_k^{i_k}$ be the prime power factorization of n where $k \geq 2$. Now let $x = p_1$, $y = (p_2)(p_3)\dots(p_k)$, then both of them share some common factors with n but $x + y$ cannot share any prime factors with n. Firstly p_1 does not divide $x + y$ since it does not divide y the other primes factors of n, i.e., $p_2, p_3, \dots, p_k$, also cannot divide $x + y$ since all of them does not divide x hence a contradiction.

7. There is more than one bus routes in a town. Every two bus routes have only one common station and every two stations are connected by a bus route.

(a) Find the number of bus routes if every route has just 3 stations.

(b) Find the number of stations on every bus route if the number of routes is 13 and every route has at least 3 stations.

(c) If every station is a vertex of a regular polygon, prove that in case (a) each route can be represented by scalene triangle and that in case (b) each bus route can be represented by a polygon such that the lengths of the segments whose end points are vertices of the polygon (representing the bus route) are all different.

No solution was received for this problem. The following solution is due to the editor.

Form an incidence matrix of stations $S_1, \ldots, S_n$ against bus routes $R_1, \ldots, R_m$ where the (i,j)-entry is 1 if Station S_i is on Route R_j and is 0 otherwise. For any two rows of this matrix, a pair of 1's that occur in the same column is called a column pair (with respect to these two rows). A row pair is defined analogously. In this matrix, there is at least 1 column pair with respective to any pair of rows and exactly one row pair with respect to any pair of columns. The row pair condition implies that there is at most one column pair with respect to any pair of rows. Thus there is also exactly one column pair with respect to any pair of rows.

	R_1	R_2	$\cdots$	R_{a_1}		$\cdots$	R_{a_2}		$\cdots$		$\cdots$
S_1	1	1	$\cdots$	1	0	$\cdots$	0	0	$\cdots$	0	$\cdots$
S_2	1	0	$\cdots$	0	1	$\cdots$	1	0	$\cdots$	0	$\cdots$
S_3	1	0	$\cdots$	0	0	$\cdots$	0	1	$\cdots$	1	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	
S_i	0	$\boxed{X_1}$			$\boxed{X_2}$			$\boxed{X_3}$			$\cdots$

Suppose column 1 has k 1's. Assume that these occur at the first k entries of R_1. By the row pair (or column pair condition) condition, in the submatrix formed by the first k rows, then each column other than the first has exactly one 1 and each row has at least two 1's. Let the 1's, other than those in the first column, occur at the following cells:

$$(1,2), \ldots, (1,a_1),$$
$$(2,a_1+1), \ldots, (2,a_2),$$
$$\ldots$$
$$(k,a_{k-1}+1), \ldots, (k,n)$$

Now consider row i, $i > k$. There is exactly one 1 in each of the sets of cells $X_1 = \{(i,2), \ldots, (i,a_1)\}$, $X_2 = \{(i,a_1+1), \ldots, (i,a_2)\}$, $\cdots$, $X_k = \{(i,a_{k-1}+1), \ldots, (i,n)\}$. Thus there are exactly k 1's in row i. By repeating the argument for other columns, we see that each of the first k rows also has exactly k 1's. By symmetry, we see that each column also has exactly k 1's. Thus $m = n = k(k-1)+1$.

(a) This is the case $k = 3$. We get $m = n = 7$.

(b) This is the case $n = 13$. We get $k = 4$.

(c) Label the vertices in the clockwise order $a_1, a_2, \ldots, a_n$. In case (a), represent one of the routes by $\Delta a_1a_2a_4$. The rest can then be obtained by rotating this triangle by $360°/7$ about the centre. In case (b), rotate the quadrilateral $a_1a_2a_5a_7$

8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(x) - yf(y) = (x-y)f(x+y) \quad \text{for any} \quad x, y \in \mathbb{R}.$$

Solution by Meng Dazhe (Raffles Junior College).

The answer is all functions in the form $f(x) = ax + b$. Let $g(x) = f(x) - f(0)$. Then $g(0) = 0$. Now let $h(x) = g(x) - xg(1)$. Thus $h(1) = 0$ and $h(0) = 0$. Also

$$xh(x) - yh(y) = (x-y)h(x+y) \tag{1}$$

We now show that $h(x) = 0$ for all $x \in \mathbb{R}$. Firstly, substituting y as $-x$ in (1)

$$xh(x) + xh(-x) = 2xh(0) = 0.$$

Hence for $x \neq 0$, $h(x) = -h(-x)$. Thus $h(-1) = 0$. Now suppose there is any number u such that $h(u) \neq 0$. Then $u \neq -1, 0, 1$. Substitute $x = u$ and $y = 1$ into (1), we get

$$uh(u) = (u-1)h(u+1) \tag{2}$$

Hence $h(u+1) \neq 0$. Substitute $x = u+1$ and $y = -1$ into (1), we get

$$(u+1)h(u+1) = (u+2)h(u).$$

Thus $u+2 \neq 0$ for otherwise $h(u+1) = 0$. Hence $u/(u-1) = (u+2)/(u+1)$ or $1 + 1/(u-1) = 1 + 1/(u+1)$ which have no solutions. Hence such u does not exist. Thus $h(x) = 0$ for all x. Substituting back, we get:

$$\begin{aligned} f(x) &= g(x) + f(0) = h(x) + xg(1) + f(0) \\ &= x[f(1) + f(0)] + f(0) \end{aligned}$$

where $f(0)$ and $f(1)$ can take any real value. Hence $f(x) = ax + b$, which is verified to work.

9. Let I be the centre of the incircle of the nonisosceles triangle ABC. The incircle touches the sides BC, CA, AB at the points A_1, B_1, C_1, respectively. Prove that the centres of the circumcircles of $\triangle AIA_1$, $\triangle BIB_1$, $\triangle CIC_1$ are collinear.

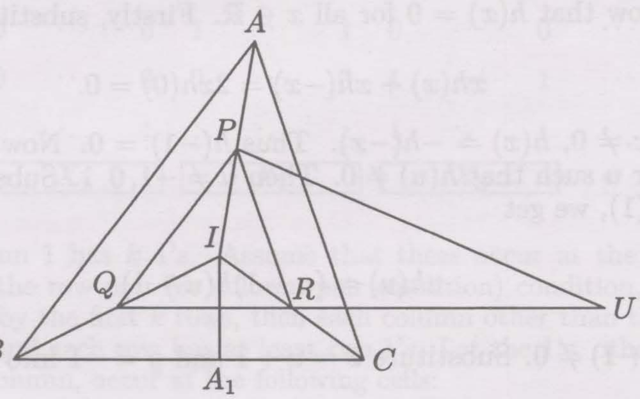
We present the solution by David Pargeter (England). Tan Kiat Chuan (Raffles Junior College) also has a similar solution.

If the bisector of the exterior angle at P on $\triangle PQR$ meets QR produced at U, then $QU/RU = PQ/PR$. This is a well-known fact and can be proved by using the sine rule.

Consider the figure of the problem, let P, Q, R be the mid-points of AI, BI, CI, respectively and define U as above. It is then easy to see that (1) PU is the perpendicular bisector of AI, (2) QR is the perpendicular bisector of IA_1. Then U is the circumcentre of $\triangle AIA_1$. Let V, W be the circumcentres of $\triangle BIB_1$, CIC_1, respectively (for simplicity, not shown in the figure), arrived in the same way. Then

$$\frac{QU}{RU}\frac{RV}{PV}\frac{PW}{QW} = \frac{PQ}{PR}\frac{RP}{RQ}\frac{QR}{QP} = 1,$$

whence the collinearity of U, V, W follows by the converse of Menelause theorem to $\triangle PQR$.



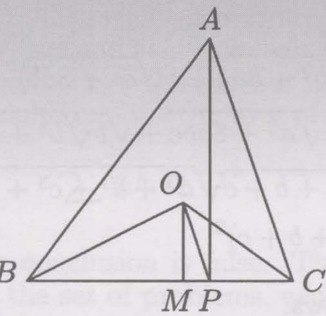
42nd International Mathematical Olympiad

Washington DC, United States of America, July 2001

1. Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A.

Suppose that $\angle BCA \geq \angle ABC + 30°$.

Prove that $\angle CAB + \angle COP < 90°$.



Essentially, the trick is to convert this to a trigonometry inequality. There are many ways to do it, we present the simplest one. Most of the other solutions involve proofing $PB \geq 3PC$ from the desired result follows readily.

Solution: Let R be the circumradius. Then

$$\begin{aligned} CP &= AC\cos C = 2R\sin B\cos C \\ &= R(\sin(B+C) - \sin(C-B)) \leq R(1-\sin(C-B)) \\ &\leq R(1-\sin 30^\circ) = R/2. \end{aligned}$$

So, $OP > OC - PC \geq PC$, and whence $\angle PCO > \angle POC$. The desired result then follows from the fact that $\angle PCO + \angle CAB = 90^\circ$.

2. Prove that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$$

for all positive real numbers a, b and c.

First solution. Note that $f(x) = \frac{1}{\sqrt{x}}$ is convex for positive x. Recall weighted Jensen's inequality:-

$$af(x) + bf(y) + cf(z) \geq (a+b+c)f(ax+by+cz).$$

Apply this to get

$$\text{LHS} \geq \sqrt{\frac{(a+b+c)^3}{a^3+b^3+c^3+24abc}} \geq 1.$$

The last step follows because by the AM-GM inequality, we have

$$(a+b+c)^3 \geq a^3+b^3+c^3+24abc.$$

Second solution. By Cauchy -Schwarz inequality we have

$$\text{LHS} \times \left(a\sqrt{a^2+8bc} + b\sqrt{b^2+8ac} + c\sqrt{c^2+8ab}\right) \geq (a+b+c)^2$$

and

$$\begin{aligned}(a\sqrt{a^2+8bc}+b\sqrt{b^2+8ac}+c\sqrt{c^2+8ab})\\ &= \sqrt{a}\sqrt{a^3+8abc}+\sqrt{b}\sqrt{b^3+8abc}+\sqrt{c}\sqrt{c^3+8abc}\\ &\le \sqrt{a+b+c}\sqrt{a^3+b^3+c^3+24abc}\\ &\le (a+b+c)^2.\end{aligned}$$

The inequality thus follows.

Third and the official solution. First we shall prove that

$$\frac{a}{\sqrt{a^2+8bc}} \ge \frac{a^{4/3}}{a^{4/3}+b^{4/3}+c^{4/3}},$$

or equivalently, that

$$(a^{4/3}+b^{4/3}+c^{4/3})^2 \ge a^{2/3}(a^2+8bc),$$

or equivalently, that

$$b^{4/3}+c^{4/3}+2a^{4/3}b^{4/3}+2a^{4/3}c^{4/3}+2b^{4/3}c^{4/3} \ge 8a^{2/3}bc.$$

The last inequality follows from the AM-GM inequality. Similarly, we have

$$\frac{b}{\sqrt{b^2+8ac}} \ge \frac{b^{4/3}}{a^{4/3}+b^{4/3}+c^{4/3}}, \qquad \frac{c}{\sqrt{c^2+8ab}} \ge \frac{c^{4/3}}{a^{4/3}+b^{4/3}+c^{4/3}}.$$

The result then follows by adding these three inequalities.

3. Twenty-one girls and twenty-one boys took part in a mathematical contest.

> Each contestant solved at most six problems.
>
> For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

Note. One useful way to investigate this problem is to form an incidence matrix. Let $B_1, B_2, \ldots, B_{21}$ be the boys and $G_1, \ldots, G_{21}$ be the girls and $P_1, \ldots, P_n$ be the problems. Set up an incidence matrix with the columns indexed by the problems and the rows indexed by the students. The entry at (S, P_i) is 1 if S solves P_i and 0 otherwise. We present two solutions based on this incidence matrix.

First solution. Let b_i be the number boys who solve P_i and g_i be the number of girls who wolve P_i. Then the number of ones in every row is at most 6. Thus $\sum_{i=1}^n b_i \le 6|B|$ and $\sum_{i=1}^n g_i \le 6|G|$.

In this matrix the rows B_i and G_j have at least a pair of ones in the same column because every boy and every girl solve a common problem. Call such a pair of ones a one-pair. Thus the number of one-pairs is at least 21^2. However, counting by the columns, the number of one-pairs is $\sum b_i g_i$. Thus we have

$$\sum g_i b_i \geq 21^2.$$

Now suppose that the conclusion is false. Then $b_i \geq 3$ implies $g_i \leq 2$ and vice versa. Let P_G be the set of problems, each of which is solved by at least 3 girls and at most 2 boys, P_B be the set of problems, each solved by at least 3 boys and at most 2 girls and P_X be the set of problems, each of which is solved by at most 2 boys and at most 2 girls. Thus

$$\sum b_i g_i = \sum_{P_i \in P_B} b_i g_i + \sum_{P_i \in P_G \cup P_X} b_i g_i \leq 2 \sum_{P_i \in P_B} b_i + 2 \sum_{P_i \in P_G \cup P_X} g_i.$$

Now for any girl G_i, consider the matrix M_i with whose columns correspond to problems solved by G_i and whose rows are all the boys. Then in this matrix, every row has at least a one. Thus there are at least 21 ones in this matrix. By the pigeonhole principle, there is a column, say P_j with at least 4 ones. Thus each girl solves at least one problem in P_B. Hence $\sum_{P_i \in P_B} g_i \geq |G|$ or equivalently, $\sum_{P_i \in P_G \cup P_X} g_i \leq 5|G|$. Similarly, $\sum_{P_i \in P_B} b_i \leq \sum_{P_i \in P_B \cup P_X} b_i \leq 5|B|$. Thus we have

$$21^2 \leq \sum b_i g_i \leq 10(|G| + |B|) = 420$$

a contradiction.

Second solution. With the same notation as in the first solution, divide the incidence matrix M into two part: M_B which is formed by the columns in $P_B \cup P_X$ and M_G which is formed by the columns in P_G. The matrix M has 441 one-pairs. Thus one of these two submatrices, say M_B, has at least 221 one-pairs. (The case for M_G foolows by symmetry.) Thus one of the girls, say G_1, contributes at least 11 one-pairs in M_B. Since each one in row G_1 contributes at most 2 one-pairs in M_B, there are 6 ones in row G_1 in M_B. This means the row G_1 in M_G does not have any ones. Thus G_1 contributes at most 12 one-pairs in M. But G_1 should contribute at least 21 one-pairs and we have a contridiction.

Third solution. Suppose each problem P_i is solved by g_i girls and b_i boys. Then $\sum g_i b_i \geq 21^2 = 441$ since each boy and each girl solved a common problem. We assume that the conclusion is false, i.e. $\min\{g_i, b_i\} \leq 2$. We also assume that each problem is solved by at least one boy and at least one girl. So

$$g_i + b_i \geq \frac{g_i b_i}{2} + 1.5 \quad \text{and} \quad \sum_{i=1}^{n} g_i + b_i \geq 220.5 + 1.5n.$$

Since each boy and each girl solved at most 6 problems, we have $\sum g_i + b_i \leq 6 \times 21 \times 2 = 252$. From these we have $n \leq 21$.

Now consider a 21×21 grid, with one side representing girls, the other boys. Each cell in the grid is filled with the problems solved by both the corresponding boy and girl. There are at most 6 problems in each row and each column and each cell must contain at least one problem. In each row R_i there is problem P_i that appears at least three times. Similarly, each column C_j has such a problem P'_j. If $P_i = P'_j$ for some i, j, then this problem is solved by three boys and three girls. So we assume that $\{P_i\}$ and $\{P'j\}$ are disjoint. Also if there exist i, j, k such that $P_i = P_j = P_k$, the this problem is solved by three girls and three boys. So the set $\{P_i\}$ contains at least 11 problems. Similarly, the set $\{P'_j\}$ contains at least 11 problems. Thus there are at least 22 problems, a contradiction.

4. Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i.$$

Prove that there are two permutations b and c, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

Official solution. This is the standard double counting argument. Compute the sum $\sum S(a)$, over all permutations. For each $i = 1, 2, \ldots, n$, the term $k_j i$, $j = 1, 2, \ldots, n$, appears $(n-1)!$ times. Thus its contribution to $\sum S(a)$ is $(n-1)! k_j i$. Thus

$$\sum S(a) = (n-1)! \sum_i i \sum_j k_j = \frac{(n+1)!}{2} \sum_j k_j \qquad (*)$$

Now suppose that the conclusion is false. Then the set $\{S(a)\}$ is a complete set of residues mod $n!$. Thus

$$\sum S(a) \equiv 1 + 2 + \cdots + n! = \frac{(n!+1)n!}{2} \equiv \frac{n!}{2} \not\equiv 0 \pmod{n!}.$$

But from $(*)$, we have $\sum S(a) = n![(n+1)/2] \sum k_j \equiv 0 \pmod{n!}$. (Note $(n+1)/2$ is an integer as n is odd.) Thus we have a contradiction.

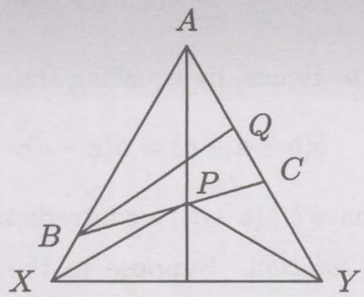
5. In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA.

It is known that $\angle BAC = 60°$ and that $AB + BP = AQ + QB$.

What are the possible angles of triangle ABC?

Solution. Extend AB to X such that $BX = BP$. Similarly, let Y be the point on AC (extended if necessary) on the opposite side of Q as A such that $BQ = QY$. Since $AB + BP = AQ + QB$, this implies that $AX =$

AY by construction, and hence ΔAXY is equilateral with AP being the perpendicular bisector of XY.



We consider first the case where Y does not coincide with C and lies on AC extended (as in the figure). Let $\angle ABQ = \angle CBQ = x$. Then since $BX = BP$, $\angle BXP = \angle BPX = x$. Also, $\angle BQC = 60° + x$ and $BQ = QY$ imply that $\angle QBY = \angle QYB = 60° - \frac{x}{2}$, so $\angle PBY = 60° - \frac{3x}{2}$. Since AP is the perpendicular bisector of XY, $\angle PXY = \angle PYX$, so that $\angle PYC = \angle PXB = x$. Thus, $\angle PYB = \angle QYB - x = 60° - \frac{3x}{2}$. Hence $\angle PBY = \angle PYB$ and $PB = PY = PX$, which implies that ΔPBX is equilateral and $x = 60°$. However, this is a degenerate case since $\angle BAC = 60°$ and $\angle ABC = 2x = 120°$. The case where Y does not coincide with C and lies in the interior of AC is similar, except that this time $\angle PBY = \angle PYB = \frac{3x}{2} - 60°$. We once again reach the conclusion that ΔPBX is equilateral and $x = 60°$, so this is a degenerate case once again.

This leaves just one case to consider where Y coincides with C. In this case, $BQ = QC$ and so $\angle ABQ = \angle CBQ = \angle BCQ = \frac{180°-60°}{3} = 40°$. We can verify that this 40°-60°-80° triangle verifies the condition of the question: Extend AB to X so that $BX = BP$. Then ΔAPX is congruent to ΔAPC, since $\angle PXB = \angle ACB = 40°$, $\angle BAP = \angle CAP = 30°$ and AP is a common side. It follows that $PX = PC$ and so $\angle PXC = \angle PCX = 20°$. Hence, $\angle AXC = \angle ACX = 60°$, so ΔAXC is equilateral. Thus, $AX = AC \Rightarrow AB + BX = AQ + QY \Rightarrow AB + BP = AQ + QB$. QED.

6. Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

Prove that $ab + cd$ is not prime.

First solution. Write the original condition as

$$a^2 - ac + c^2 = b^2 + bd + d^2 \qquad (*)$$

Assume that $ab + cd = p$ is prime. Then $a = (p - cd)/b$. Substituting this into $(*)$, we get

$$p(ab - cd - cb) = (b^2 - c^2)(b^2 + bd + d^2).$$

since $1 < b^2 - c^2 < ab < p$, we have $p \mid (b^2 + bd + d^2)$. But

$$b^2 + bd + d^2 < 2ab + cd < 2p,$$

we have $b^2 + bd + d^2 = p$. Hence, by equating the expressions for p, we get

$$b(b + d - a) = d(c - d).$$

Since $\gcd(b, d) = 1$, we have $b \mid (c - d)$, a contradiction because $0 < c - d < b$.

Second and the official solution. Suppose to the contrary that $ab + cd$ is prime. Note that

$$ab + cd = (a + d) + (b - c)a = m \cdot \gcd(a + d, b - c)$$

for some positive integer m. By assumption, either $m = 1$ or $\gcd(a+d, b-c) = 1$.

Case (i): $m = 1$. Then

$$\begin{aligned}\gcd(a + d, b - c) &= ab + cd > ab + cd - (a - b + c + d)\\ &= (a + d)(c - 1) + (b - c)(a + 1)\\ &\geq \gcd(a + d, b - c).\end{aligned}$$

which is false.

Case (ii): $\gcd(a+d), b-c) = 1$. Substituting $ac + bd = (a + d)b - (b - c)a$ for the left hand side of $a + c + bd = (b + d + a - c)(b + d - a + c)$, we obtain

$$(a + d)(a - c - d) = (b - c)(b + c + d).$$

In view of this, there exists a positive integer k such that

$$\begin{aligned}a - c - d &= k(b - c),\\ b + c + d &= k(a + d).\end{aligned}$$

Adding we get $a + b = k(a + b - c + d)$ and thus $k(c - d) = (k - 1)(a + b)$. Recall that $a > b > c > d$. If $k = 1$ then $c = d$, a contradiction. If $k \geq 2$ then

$$2 \geq \frac{k}{k - 1} = \frac{a + b}{c - d} > 2,$$

a contradiction.