COUNTING – Its Principles & Techniques (11)

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31. A More Challenging Shortest Route Problem

We have learnt in Section 6 [21] that the number of shortest $P-Q$ routes in the 4×2 rectangular grid of Fig. 31.1 is, by (*BP*), equal to the number of 6-digit binary sequences with 4 0's and 2 1's, which is $\binom{6}{2}$.

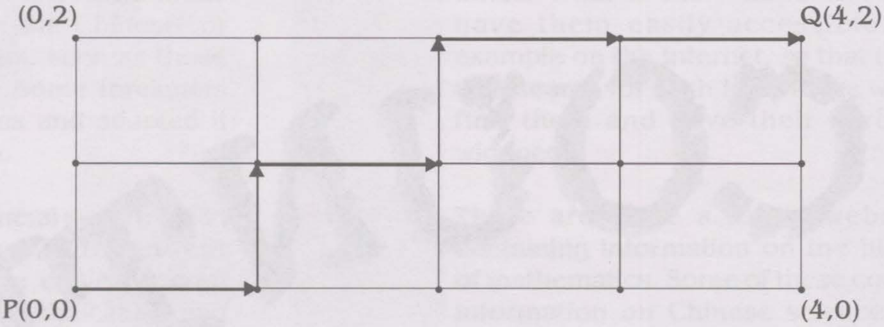


Fig. 31.1

In general, in the rectangular coordinate system of Fig. 31.2, the number of shortest routes from $P(a,b)$ to $Q(c,d)$, where a, b, c, d are integers with $a \le c$ and $b \le d$, is given by

$$\binom{(c+d)-(a+b)}{c-a} \quad \text{or} \quad \binom{(c+d)-(a+b)}{d-b} \qquad (31.1)$$

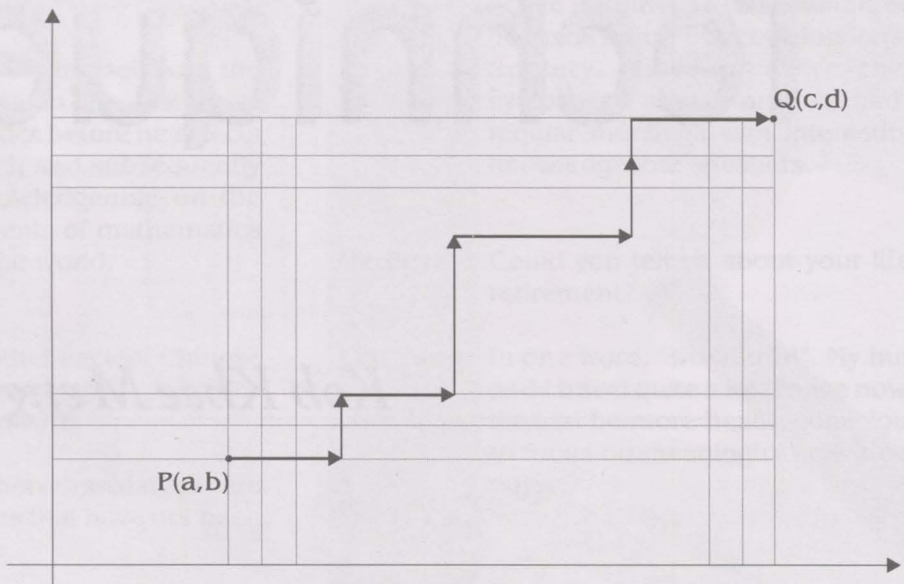
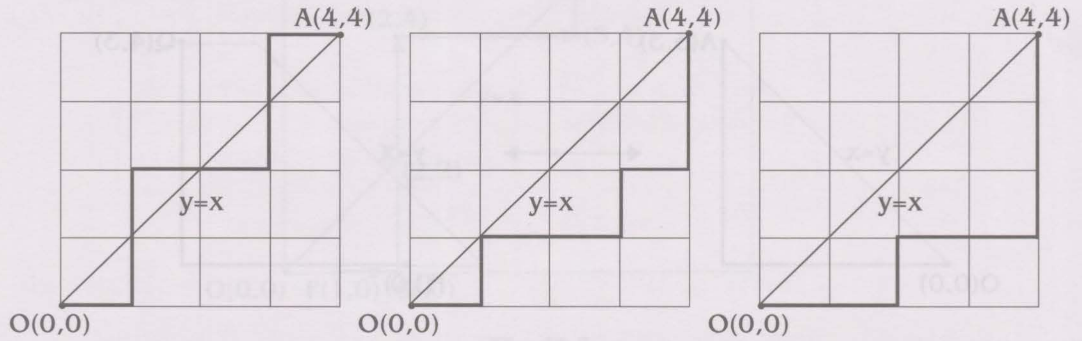


Fig. 31.2

Consider the case when $O=(0,0)$ and $A=(n,n)$, where n is a positive integer. By (31.1), the number of shortest $O-A$ routes is given by $\binom{2n}{n}$. As shown in Fig. 31.3 (where $n=4$), we observe that the $\binom{2n}{n}$ shortest $O-A$ routes can be divided into two groups : those that cross the diagonal $y=x$ (see (i)) and those that do not (see (ii) and (iii)).



(i) crossing y=x (ii) not crossing y=x but meeting y=x internally (iii) not meeting y=x internally, and so not crossing y=x

Fig. 31.3

Around 1887, the French combinatorist Désiré André (1840-1917) studied the following problem.

> **A** How may shortest routes from $O(0,0)$ to $A(n,n)$ are there which do not cross the diagonal $y=x$ in the rectangular coordinate system? (31.2)

For convenience, let us denote by $f(n)$ the number of *such* shortest routes from $O(0,0)$ to $A(n,n)$. For $n=1,2,3,$ all such routes and the values of $f(n)$ are shown in Table 31.1.

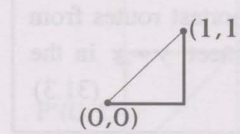
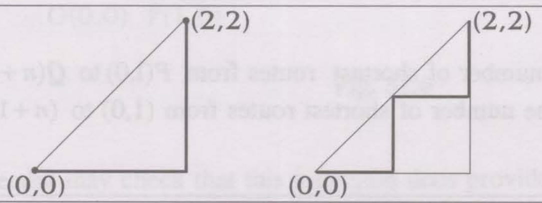
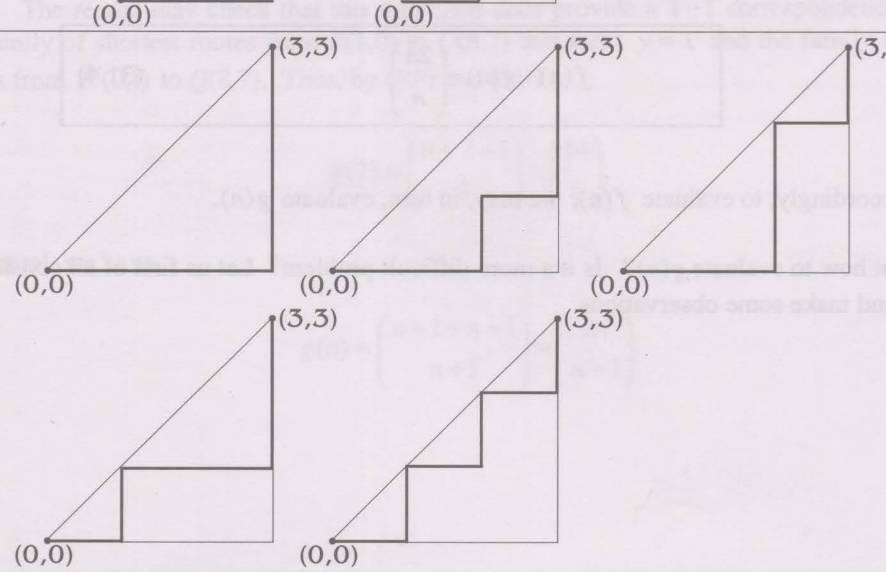
n	the desired routes	*f*(*n*)
1		1
2		2
3		5

Table 31.1

In what follows, we shall present André's elegant idea in solving the problem.

By translating a route in the coordinate system one unit to the right as shown in Fig. 31.4, we see that there is a $1-1$ correspondence between the family of shortest routes from $O(0,0)$ to $A(n,n)$ that do not *cross* $y=x$ and the family of shortest routes from $P(1,0)$ to $Q(n+1,n)$ that do not *meet* $y=x$.

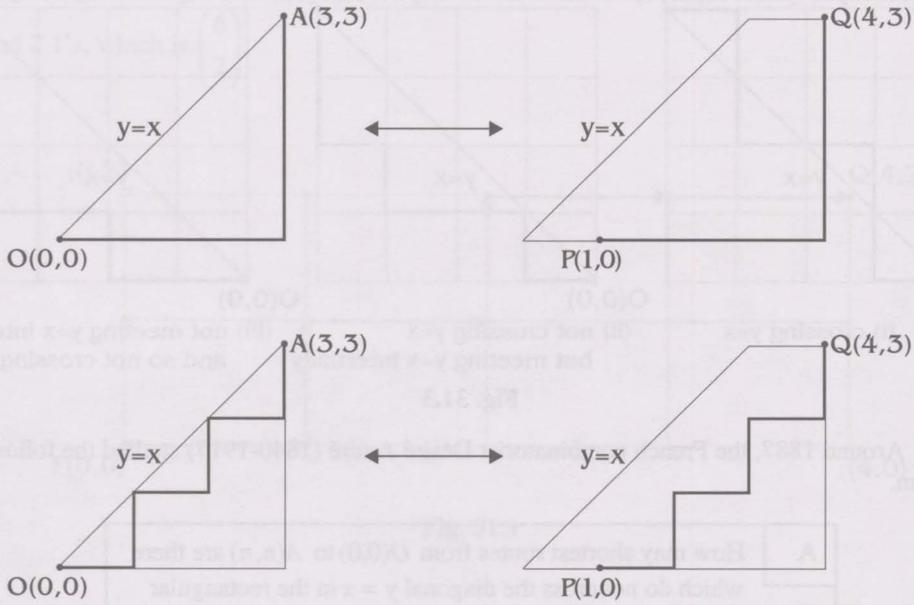


Fig. 31.4

Thus, by (*BP*), we have:

> $f(n)$ is equal to the number of shortest routes from $P(1,0)$ to $Q(n+1,n)$ that do not meet $y=x$ in the coordinate system. (31.3)

Now, let $g(n)$ denote the number of shortest routes from $P(1,0)$ to $Q(n+1,n)$ that *meet* $y=x$. Clearly, $f(n)+g(n)$ is the number of shortest routes from $(1,0)$ to $(n+1,n)$. Thus, by (31.1), we have:

$$f(n)+g(n)=\binom{2n}{n}. \tag{31.4}$$

Accordingly, to evaluate $f(n)$, we may, in turn, evaluate $g(n)$.

But how to evaluate $g(n)$? Is it a more difficult problem? Let us first of all consider an example and make some observations.

Fig. 31.5 shows a shortest route from $P(1,0)$ to $Q(8,7)$ (here, $n = 7$) that meets $y = x$.

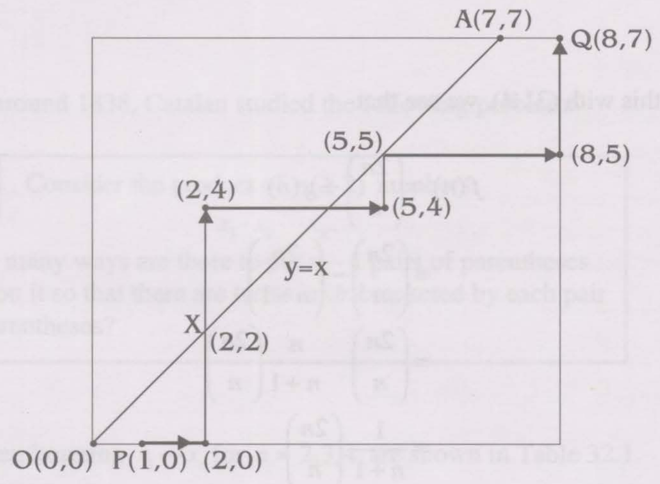


Fig. 31.5

Imagine that we are now traversing the route from P to Q. Let X be the point where the route meets $y = x$ for the first time (in Fig 31.5, $X = (2,2)$; note that such a X always exists). Consider the reflection of this part of the route from P to X with respect to $y = x$ as shown in Fig 31.6. Beginning with this image of reflection and following the rest of the original shortest route from X to Q, we obtain a shortest route from $P'(0,1)$ to $Q(8,7)$.

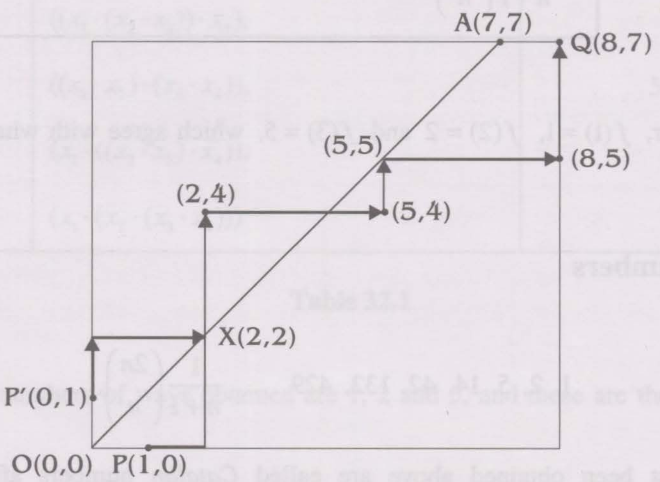


Fig. 31.6

The reader may check that this reflection does provide a $1-1$ correspondence between the family of shortest routes from $P(1,0)$ to $Q(8,7)$ that *meet* $y = x$ and the family of shortest routes from $P'(0,1)$ to $Q(8,7)$. Thus, by (*BP*) and (31.1),

$$g(7) = \binom{8+7-1}{8} = \binom{14}{6}.$$

In general, we have

$$g(n) = \binom{n+1+n-1}{n+1} = \binom{2n}{n-1}.$$

Combining this with (31.4), we see that

$$\begin{aligned} f(n) &= \binom{2n}{n} - g(n) \\ &= \binom{2n}{n} - \binom{2n}{n-1} \\ &= \binom{2n}{n} - \frac{n}{n+1}\binom{2n}{n} \\ &= \frac{1}{n+1}\binom{2n}{n}. \end{aligned}$$

That is:

> the number of shortest routes from $O(0,0)$ to $A(n,n)$ which do not cross the diagonal : $y = x$
> $= f(n)$
> $= \dfrac{1}{n+1}\dbinom{2n}{n}.$

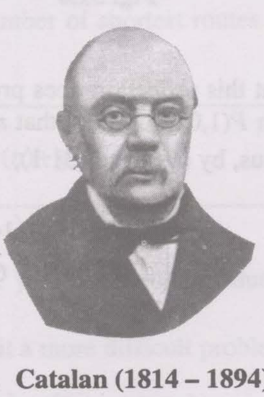
In particular, $f(1) = 1$, $f(2) = 2$ and $f(3) = 5$, which agree with what were shown in Table 31.1.

32. Catalan Numbers

The numbers:

$$1,\ 2,\ 5,\ 14,\ 42,\ 132,\ 429,\ \ldots,\ \frac{1}{n+1}\binom{2n}{n},\ \ldots$$

that have just been obtained above are called *Catalan* numbers after the Belgium mathematician Eugene Charles Catalan.



Catalan (1814 – 1894)

Indeed, around 1838, Catalan studied the following problem:

> **B** Consider the product of $n(\geq 2)$ numbers:
>
> $$x_1 \cdot x_2 \cdots x_n.$$
>
> How many ways are there to put $n-1$ pairs of parentheses '(,)' on it so that there are terms $a \cdot b$ bracketed by each pair of parentheses?

The ways of parenthesizing $x_1 \cdots x_n$ for $n = 2,3,4$ are shown in Table 32.1.

n	$x_1 \cdot x_2 \cdot \cdots \cdot x_n$	number of ways
2	$(x_1 \cdot x_2)$	1
3	$((x_1 \cdot x_2) \cdot x_3), (x_1 \cdot (x_2 \cdot x_3))$	2
4	$(((x_1 \cdot x_2) \cdot x_3) \cdot x_4)$, $((x_1 \cdot (x_2 \cdot x_3)) \cdot x_4)$, $((x_1 \cdot x_2) \cdot (x_3 \cdot x_4))$, $(x_1 \cdot ((x_2 \cdot x_3) \cdot x_4))$, $(x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)))$	5

Table 32.1

It turns out the numbers of ways obtained are 1, 2 and 5, and these are the first three Catalan numbers.

Note. It was reported in [22] that the Catalan sequence was found and studied by a Mongolian mathematician Ming An-Tu (1692-1763) in the 18th century.

Let us proceed to present another problem which is equivalent to the one introduced by Just [19]:

> **C** For each positive integer n, how many $2n$-digit binary sequences $b_1 b_2 \cdots b_{2n}$ with n 0's and n 1's are there such that for each $i = 1,2,\ldots,2n$, the number of 0's is larger than or equal to the number of 1's in the subsequence $b_1 b_2 \cdots b_i$?

Table 32.2 shows all such binary sequences for $n = 1,2,3$. Notice that the numbers of such sequences are again the first three Catalan numbers.

n	$b_1 b_2 \cdots b_{2n}$	number of such sequences
1	01	1
2	0011, 0101	2
3	000111, 001011, 001101, 010011, 010101	5

Table 32.2

The solution of Problem (A) given by André gives rise to the Catalan numbers: $\frac{1}{n+1}\binom{2n}{n}$. The answers of the first three initial cases for Problems (B) and (C) are 1, 2 and 5, which are Catalan numbers. Is it true that the answers of Problems (B) and (C) for general cases are also Catalan numbers?

Yes, they are! In Table 32.3, we exhibit by examples 1 – 1 correspondences among the routes for Problem (A), the ways of parenthesizing $x_1 \cdots x_n$ for Problem (B) and the binary sequences for Problem (C); and we leave it to the reader to figure out the rules of the correspondences.

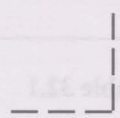
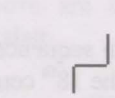
Problem (C) ($n = 3$)	Problem (A) ($n = 3$)	Problem (B) ($n = 4$)
000111		$(((x_1 \cdot x_2) \cdot x_3) \cdot x_4)$
001011		$((x_1 \cdot (x_2 \cdot x_3)) \cdot x_4)$
001101		$((x_1 \cdot x_2) \cdot (x_3 \cdot x_4))$
010011		$(x_1 \cdot ((x_2 \cdot x_3) \cdot x_4))$
010101		$(x_1 \cdot (x_2 \cdot (x_3 \cdot x_4)))$

Table 32.3

One of the more general problems of this type, known as the Ballot Problem, is stated below.

> **The Ballot Problem** X and Y are the two candidates taking part in an election. Assume that at the end X receives x votes and Y receives y votes with $x > y$ (and so X wins). What is the probability that X always stays ahead of Y throughout the counting of the votes?

To find out the desired probability, the essential part of the solution is to find out the number of ways that the ballots are proceeded in such a way that when they are counted one at a time the number of votes for X is always more than that for Y. This problem is clearly an extension of Problems (A) and (C). Employing the ideas and techniques used to solve Problem (A), André solved in 1887 this more general problem. Indeed, the Ballot Problem was first posed and solved by Joseph Louis Francois Bertrand (1822-1900) in the same year, i.e., 1887. The reader may refer to [12] for more details and to [2] for the history and some generalizations of the problem.

It was said that in 1751, the Swiss mathematician Leonard Euler (1707-1783) proposed to Christian Goldbach (1690-1764) the following famous problem, which was later solved by Johann Andreas von Segner (1704-1777) in 1758 and by Catalan in 1838 using different methods.

> **(D) Euler's Polygon Division Problem** A *triangulation* of an n-sided polygon, where $n \geq 3$, is a subdivision of the polygon into triangles by means of its nonintersecting diagonals. How many different triangulations are there of an n-sided polygon?

All the triangulations of an n-sided polygon, where $n = 3, 4, 5$, are shown in Table 32.4. The reader may notice that the respective numbers of triangulations are the first three Catalan numbers.

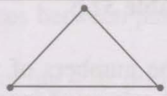
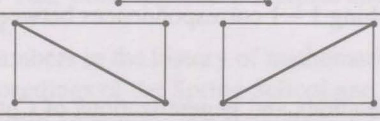
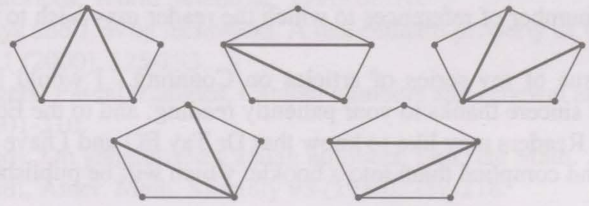
n	the triangulations	number of triangulations
3		1
4		2
5		5

Table 32.4

Finally, let us introduce another interesting problem.

E	There are $2n\,(n \geq 1)$ distinct fixed points on the circumference of a circle. How many ways are there to pair off them by n nonintersecting chords?

Table 32.5 shows all the ways for $n = 1, 2, 3$. Again, the number of ways are the first three Catalan numbers.

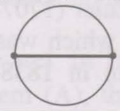
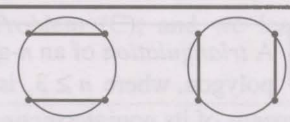
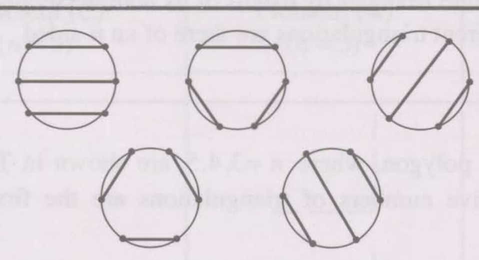
n	pairing off $2n$ points on the circumference of a circle	number of ways
1		1
2		2
3		5

Table 32.5

The reader is invited to show that the numbers of ways for Problems (D) and (E) are indeed Catalan numbers by establishing 1 – 1 correspondences between Problem (D) (resp., (E)) and any of Problems (A) – (C).

For more examples, interpretations and generalizations of Catalan numbers, we include at the end of this article a number of references to which the reader may wish to refer.

This is the last issue of my series of articles on Counting. I would like to take this opportunity to express my sincere thanks to your patiently reading; and to the Editorial Board of Medley for their support. Readers may like to know that Dr Tay EG and I have revised the first six articles of this series and complied them into a booklet which will be published soon.

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