COMPETITION CORNER

In this issue we publish the problems of the Mathematical Competitions in Croatia 2000, Bulgarian Mathematical Olympiad 1994, and the 42nd International Mathematical Olympiad which was held in Washington DC, United States of America, July 2001.

Please send your solutions of these Olympiads to me at the address given. All correct solutions will be acknowledged. We also present solutions of First Hong Kong (China) Mathematical Olympiad Contest 1999, Greek National Mathematical Olympiad 2000 and XII Asian Pacific Mathematical Olympiad, March 2000.

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• Problems •

Mathematical Competitions in Croatia 2000

Selected problems

1. Find all integer solutions of the equation

$$\frac{1}{x} + \frac{2}{y} - \frac{3}{z} = 1.$$

2. The incircle of $\triangle ABC$ touches its sides BC, CA, and AB in the points A_1, B_1 and C_1, respectively. Determine the angles of $\triangle A_1B_1C_1$ in terms of angles of $\triangle ABC$.

3. Let $ABCD$ be a square with side length 20. Let T_i, $i = 1, 2, \ldots, 2000$, be points in its interior so that no three points from the set $S = \{A, B, C, D\} \cup \{T_i : i = 1, 2, \ldots, 2000\}$ are collinear. Prove that at least one triangle with vertices in S has area less than $\frac{1}{10}$.

4. The circle with centre on the base BC of an isosceles triangle ABC is tangent to equal sides AB, and AC. Let P and Q be points on the sides AB and AC, respectively. Prove that

$$PB \cdot CQ = \frac{BC^2}{4}$$

if and only if PQ is tangent to his circle.

5. Let $n(\geq 3)$ positive integers be written on a circle so that each of them divides the sum of its neighbours. Denote

$$S_n = \frac{a_n + a_2}{a_1} + \frac{a_1 + a_3}{a_2} + \cdots \frac{a_{n-2} + a_n}{a_{n-1}} + \frac{a_{n-1} + a_1}{a_n}.$$

Determine the maximum and minimum of S_n.

6. Let $S = \{k \in \mathbb{N} : a \in \mathbb{N}, a^2 \mid k \Rightarrow a = 1\}$. For any $n \in \mathbb{N}$, prove that

$$\sum_{k \in S} \lfloor \sqrt{n/k} \rfloor = n.$$

Note: For any real number x, $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Bulgarian Mathematical Olympiad, 1994

Selected problems from competitions of various levels.

1. Thirty-three natural numbers are given. The prime divisors of each of the numbers are among $2, 3, 5, 7, 11$. Prove that the product of two of the numbers is a perfect square.

2. Let

$$f(x) = x^4 - 4x^3 + (3 + m)x^2 - 12x + 12$$

where m is a real number.

(a) Find all integers m such that the equation $f(x) - f(1 - x) + 4x^3 = 0$ has at least one integer solution.

(b) Find all values of m such that $f(x) \geq 0$ for all real number x.

3. Let N_0 be the set of nonnegative integers and $f(n)$ is a function $f : N_0 \to N_0$ such that $f(f(n)) + f(n) = 2n + 3$ for every $n \in N_0$. Evaluate $f(1993)$.

4. A convex quadrilateral $ABCD$ is inscribed in a circle with centre O and diameter 25. P and Q are points on AD and CD, respectively, such that $OP \perp AD$ and $OQ \perp CD$. Find the lengths of the sides of $ABCD$ if the lengths of AB, BC, CD, DA, OP, OQ are distinct natural numbers.

5. A point D lies on the side AB of $\triangle ABC$. The excircle k_1 of $\triangle ACD$, which touches the side CD externally, touches the sides AC and AD at points P and L, respectively. The excircle k_2 of $\triangle BCD$, which touches the side CD externally, touches the sides BC and BD at points Q and K, respectively. The incircle k_3 of $\triangle ACD$ touches the sides AC and AD at the points M and E, respectively and the incircle k_4 of $\triangle BCD$ touches the sides BC and BD at the points N and F, respectively.

(a) Prove that $FK = EL = MP = NQ$.

(b) If $\angle ACB = 90°$ determine the position of the point D so that the area of the convex quadrilateral $MNPQ$ is minimal.

6. Let $n > 1$ be a natural number and

$$A_n = \{x \in \mathbb{N} : \gcd(x, n) \neq 1\}.$$

The number n is called *interesting* if for any $x, y \in A_n$, we have $x + y \in A_n$. Find all interesting n.

7. There is more than one bus routes in a town. Every two bus routes have only one common station and every two stations are connected by a bus route.

(a) Find the number of bus routes if every route has just 3 stations.

(b) Find the number of stations on every bus route if the number of routes is 13 and every route has at least 3 stations.

(c) If every station is a vertex of a regular polygon, prove that in case (a) each route can be represented by scalene triangle and that in case (b) each bus route can be represented by a polygon such that the lengths of the segments whose end points are vertices of the polygon (representing the bus route) are all different.

8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(x) - yf(y) = (x-y)f(x+y) \quad \text{for any} \quad x, y \in \mathbb{R}.$$

9. Let I be the centre of the incircle of the nonisosceles triangle ABC. The incircle touches the sides BC, CA, AB at the points A_1, B_1, C_1, respectively. Prove that the centres of the circumcircles of $\triangle AIA_1$, $\triangle BIB_1$, $\triangle CIC_1$ are collinear.

42nd International Mathematical Olympiad

Washington DC, United States of America, July 2001

1. Let ABC be an acute-angled triangle with circumcentre O. Let P on BC be the foot of the altitude from A.

Suppose that $\angle BCA \geq \angle ABC + 30°$.

Prove that $\angle CAB + \angle COP < 90°$.

2. Prove that

$$\frac{a}{\sqrt{a^2+8bc}} + \frac{b}{\sqrt{b^2+8ca}} + \frac{c}{\sqrt{c^2+8ab}} \geq 1$$

for all positive real numbers a, b and c.

3. Twenty-one girls and twenty-one boys took part in a mathematical contest.

Each contestant solved at most six problems.

For each girl and each boy, at least one problem was solved by both of them.

Prove that there was a problem that was solved by at least three girls and at least three boys.

4. Let n be an odd integer greater than 1, and let $k_1, k_2, \ldots, k_n$ be given integers. For each of the $n!$ permutations $a = (a_1, a_2, \ldots, a_n)$ of $1, 2, \ldots, n$, let

$$S(a) = \sum_{i=1}^{n} k_i a_i.$$

Prove that there are two permutations b and c, $b \neq c$, such that $n!$ is a divisor of $S(b) - S(c)$.

5. In a triangle ABC, let AP bisect $\angle BAC$, with P on BC, and let BQ bisect $\angle ABC$, with Q on CA.

It is known that $\angle BAC = 60°$ and that $AB+BP = AQ+QB$.

What are the possible angles of triangle ABC?

6. Let a, b, c, d be integers with $a > b > c > d > 0$. Suppose that

$$ac + bd = (b + d + a - c)(b + d - a + c).$$

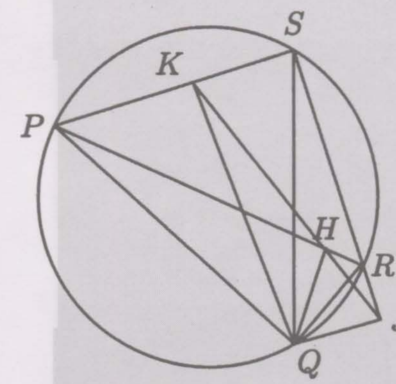
Prove that $ab + cd$ is not prime.

• Solutions •

Hong Kong (China) Mathematical Olympiad, 1999

1. $PQRS$ is a cyclic quadrilateral with $\angle PSR = 90°$; H, K are the feet of the perpendiculars from Q to PR, PS (suitably extended if necessary), respectively. Show that HK bisects QS.

Two different solutions were received. First we present the solution provided independently by Zachary Leung Ngai Hang (Anglo-Chinese School (Independent)), Meng Dazhe (River Valley High School), R. Pargeter (England) and Lu Shangyi (National University of Singapore).



Drop a perpendicular from Q to RS, meeting it at J. H, K and L are collinear as they lie on the Simpson line from Q to $\triangle PSR$. Thus $QJSH$ is a rectangle with HJ and QS as diagonals. Thus HK bisects QS.

(Note: The feet of the perpendiculars from Q to $\triangle PSR$ are collinear. The line is called the Simpson Line. This fact can be proved by considering cyclic quadrilaterals and is left to the reader.)

Next we have the solution by Tan Kiat Chuan and Nicholas Tham (Raffles Junior College) and Calvin Lin Zhiwei (Hwachong Junior College).

Let HK meet QS at X. We have $QK \parallel RS$ since they are both perpendicular to KS. Also since $\angle QHP = \angle QKP = 90°$, $QHKP$ is cyclic. Thus

$$\angle KQS = \angle QSR = \angle QPR = \angle QKH.$$

Therefore $QX = XK$. Also

$$\angle HKS = 90° - \angle QKH = 90° KQS = \angle QSK.$$

So $XK = XS$. Therefore HK bisects QS.

2. The base of a pyramid is a convex polygon with 9 sides. Each of the diagonals of the base and each of the edges on the lateral surface of the pyramid is coloured either black or white. Both colours are used. (Note that the sides of the base are not coloured.) Prove that there are three segments coloured the same colour which form a triangle.

Correct solutions were received from Meng Dazhe (River Valley High School), Nicholas Tham, Tan Kiat Chuan, Julius Poh (Raffles Junior College), Calvin Lin (Hwachong Junior College), Joel Tay Wei En, Zachary Leung Ngai Hang (Anglo-Chinese School (Independent)). We present the similar solution by Meng, Tham, Lin and Tay.

Let P be the apex of the pyramid. By the pigeonhole principle, at least 5 of the lateral sides, say PA, PB, PC, PD, PE of the pyramid are coloured with the same colour, say white. Assume that the five vertices A, B, C, D, E appear in that order at the base. Among the five edges, AB, BC, CD, DE and EA, at least one, say AB, is a diagonal. Then AB, BD and DA are all diagonals. If one of them is coloured white, then these together with P form a white triangle. Otherwise, ABD is a black triangle.

3. Let s, t be given nonzero integers, and let (x, y) be any ordered pair of integers. A move changes (x, y) to $(x+t, y-s)$. The pair (x, y) is **good** if after some (may be zero) number of moves it describes a pair of integers that are **not** relatively prime.

(a) Determine if (s, t) is a good pair.

(b) Show that for any s and t there is pair (x, y) which is not good.

Solutions by Zachary Leung Ngai Hang (Anglo-Chinese School (Independent)), Calvin Lin (Hwachong Junior College), Lu Shangyi (National University of Singapore) and Tan Kiat Chuan (Raffles Junior College). We present solution by Tay.

(a) If $\gcd(s, t) \neq 1$, then (s, t) is a good pair. Thus we suppose $\gcd(s, t) = 1$. Let $s^2 + t^2 = k$. After m moves, we get $(s + mt), t - ms)$ and

$$s(s + mt) + t(t - ms) = k. \qquad (*)$$

Since $\gcd(s, t) = 1$, $\gcd(k, s) = \gcd(k, t) = 1$. Thus there exists m' such that $m't \equiv -s \pmod{k}$. Then from $(*)$ we also have $m's \equiv t \pmod{k}$. Thus $\gcd(s + m't, t - m's) \geq k > 1$ and (s, t) is good.

(b) Let $\gcd(s, t) = d$ and $s' = s/d$, $t' = t/d$. Choose (x, y) such that $d = sx + ty$. After i moves we get $(x_i, y_i) = (x + it, y - is)$. Thus $sx_i + ty_i = sx + ty = d$ or $s'x_i + t'y_i = 1$, i.e., $\gcd(x_i, y_i) = 1$ for all i. Thus (x, y) is not good.

4. Let f be a function defined on the positive reals with the following properties:

(1) $f(1) = 1$,

(2) $f(x + 1) = xf(x)$,

(3) $f(x) = 10^{g(x)}$, where $g(x)$ is a function defined on the reals satisfying

$$g(ty + (1 - t)z)) \leq tg(y) + (1 - t)g(z)$$

for all y and z and for $0 \le t \le 1$.

(a) Prove that

$$t[g(n) - g(n-1)] \le g(n+t) - g(n) \le t[g(n+1) - g(n)]$$

where n is an integer and $0 \le t \le 1$.

(b) Prove that $\frac{4}{3} \le f(\frac{1}{2}) \le \frac{4}{3}\sqrt{2}$.

The following is the combination of solutions by Lu Shangyi (National University of Singapore), Calvin Lin (Hwachong Junior College) and Tan Kiat Chuan (Raffles Junior College).

(a) By condition (3), the function g is concave upwards. This means that if A and B are two points on the graph of $y = g(x)$, then the portion of the graph between A and B lies beneath the line AB. The first expression is the gradient of the line joining $g(n-1)$ to $g(n)$, the second is the gradient of the line joining $g(n)$ to $g(n+t)$ while the third is the gradient of the line joining $g(n)$ to $g(n+1)$. Thus the inequality follows:

$$\frac{g(n) - g(n-1)}{n - (n-1)} \le \frac{g(n+t) - g(n)}{(n+t) - n} \le \frac{g(n+1) - g(n)}{(n+1) - n}$$

for $0 \le t \le 1$.

(b) First we note that $f(2) = 1f(1) = f(1)$. Also $f(n)/f(n-1) = n-1$. From (a) we have

$$t[g(n) - g(n-1)] \le g(n+t) - g(n) \le t[g(n+t) - g(n)].$$

Since $f(x) = 10^{g(x)}$, we have $\log f(x) = g(x)$. Substituting into (a) we have

$$\begin{aligned} t[\log f(n) - \log f(n-1)] &\le \log f(n+t) - \log f(n) \\ &\le t[\log f(n+1) - \log f(n)]. \end{aligned}$$

Simplifying we get

$$\log\left(\frac{f(n)}{f(n-1)}\right)^t \le \log\left(\frac{f(n+t)}{f(n)}\right) \le \log\left(\frac{f(n+1)}{f(n)}\right)^t$$

or

$$(n-1)^t = \left(\frac{f(n)}{f(n-1)}\right)^t \le \left(\frac{f(n+t)}{f(n)}\right) \le \left(\frac{f(n+1)}{f(n)}\right)^t = n^t.$$

Let $n = 2$ and $t = 1/2$, we have

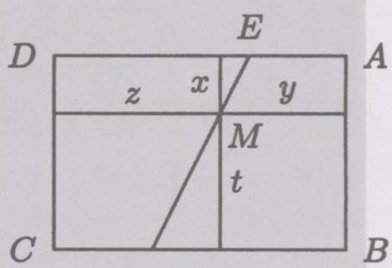
$$1 \le f(5/2)/f(2) = f(5/2) = \frac{3}{2}f(\frac{3}{2} = \frac{3}{4}f(\frac{1}{2}) \le \sqrt{2}.$$

Hence $4/3 \le f(1/2) \le 4\sqrt{2}/3$.

Greek National Mathematical Olympiad 2000

1. Consider the rectangle $ABCD$ with $AB = \alpha$, $AD = \beta$. A line ℓ passing through the centre O of the rectangle meets the side AD at the point E such that $AE/ED = 1/2$. On this line take an arbitrary point M lying inside the rectangle. Find the necessary and sufficient condition on α and β so that distances from M to the sides of the rectangle AD, AB, DC and BC, taken in that order, form an arithmetic progression.

The following is a combination of solutions by Lu Shangyi (National University of Singapore), Calvin Lin (Hwachong Junior College), R. Pargeter (England) and Joel Tay (Anglo-Chinese School (Independent)).



Let x, y, z, t be the respective distances from M to the sides AD, AB, CD, BC. If they form an arithmetic progression, then $x + t = y + z$ and hence $\alpha = \beta$ which is a necessary condition.

Now suppose that $\alpha = \beta$. Let $\theta = x/\alpha$. Note that $0 \leq \theta \leq 1$. The distances in the question are $x = \theta\alpha$, $y = (\theta + 1)\alpha/3$, $z = (2 - \theta)\alpha/3$, $t = (1 - \theta)\alpha$. These are obviously in arithmetic progression. Thus $\alpha = \beta$ is also sufficient.

2. Find the prime number p so that $1 + p^2 + p^3 + p^4$ is a perfect square, i.e. the square of an integer.

Similar solutions by Calvin Lin (Hwachong Junior College), Lu Shangyi (National University of Singapore) and Joel Tay (Anglo-Chinese School (Independent)).

Let $f(n) = 1 + n^2 + n^3 + n^4$. Indeed $f(1) = 4 = 2^2$. We'll show that for all positive integer $n > 1$, $f(n)$ is not a square. First note that

$$\begin{aligned} & (n^2 + n - 1)^2 < 1 + n^2 + n^3 + n^4 \\ \Leftrightarrow \quad & n^4 + 2n^3 - n^2 - 2n + 1 < 1 + n^2 + n^3 + n^4 \\ \Leftrightarrow \quad & n(n+1) > 0 \end{aligned}$$

which is true when $n > 0$. Also

$$\begin{aligned} & 1 + n^2 + n^3 + n^4 < (n^2 + n)^2 \\ \Leftrightarrow \quad & 1 + n^2 + n^3 + n^4 < n^4 + 2n^3 + n^2 \\ \Leftrightarrow \quad & n^3 > 1. \end{aligned}$$

Thus when $n > 1$, we have

$$(n^2 + n - 1)^2 < 1 + n^2 + n^3 + n^4 < (n^2 + n)^2.$$

Thus $f(n)$ is not a square when $n > 1$.

3. Find the maximum positive real number k such that

$$\frac{xy}{\sqrt{(x^2+y^2)(3x^2+y^2)}} \leq \frac{1}{k}$$

for all positive real numbers x and y.

Similar solutions by Joel Tay (Anglo-Chinese School (Independent)), Lu Shangyi (National University of Singapore) and Calvin Lin (Hwachong Junior College).

We have

$$\frac{xy}{\sqrt{(x^2+y^2)(3x^2+y^2)}} \leq \frac{1}{k}.$$

Thus

$$k^2 \leq \frac{(x^2+y^2)(3x^2+y^2)}{x^2y^2}.$$

Let $x^2 = a$, $y^2 = b$. Then

$$k^2 - 4 \leq \frac{3a^2+b^2}{ab}.$$

Since the above inequality must hold for all positive real numbers a, b, and $\frac{3a^2+b}{ab} \geq 2\sqrt{3}$, we have $k^2 - 4 \leq 2\sqrt{3}$.Hence the maximum value of k satisfies $k^2 = 4 + 2\sqrt{3}$ or $k = \sqrt{2(2+\sqrt{3})} = 1 + \sqrt{3}$.

4. For the subset $A_1, \ldots, A_{2000}$ of the set M, we have $|A_i| \geq 2|M|/3$, $i = 1, 2, \ldots, 2000$, where $|X|$ denotes the cardinality of the set X. Prove that there exists $\alpha \in M$ which belongs to at least 1334 from the subsets A_i.

Solution by Calvin Lin (Hwachong Junior College).

Let $M = \{a_1, a_2, \ldots, a_n\}$. Form the incidence matrix with the rows indexed by $a_1, a_2, \ldots, a_n$ and the columns indexed by $A_1, A_2, \ldots, A_{2000}$. The entry at (a_i, A_j) is 1 if $a_i \in A_j$ and is 0 otherwise. We shall count the total number of ones in the matrix in two ways. Counting by the columns, the number of ones is at least $4000n/3$. The average number of ones per row is $4000/3$. Hence there is one row with $\lceil 4000/3 \rceil = 1334$ ones. This means that the corresponding element belongs to at least 1334 of the sets.

XII Asian Pacific Mathematical Olympiad

March 2000

1. Compute the sum

$$S = \sum_{i=0}^{101} \frac{x_i^3}{1 - 3x_i + 3x_i^2}$$

for $x_i = \frac{i}{101}$.

Solution by Joel Tay (Anglo-Chinese School (Independent)) and Lu Shangyi (National University of Singapore).

Note that

$$\frac{x_i^3}{1 - 3x_i + 3x_i^2} = \frac{i^3}{101(3i^2 - 303i + 101^2)}.$$

Also if $j = 101 - i$, then

$$3j^2 - 303j + 101^2 = 3i^2 - 303i + 101^2.$$

Thus

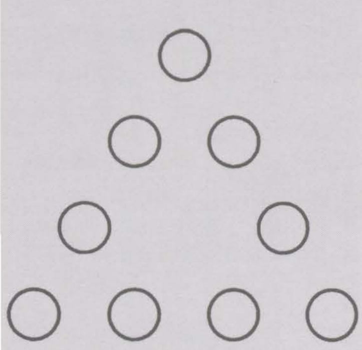
$$\frac{x_j^3}{1 - 3x_j + 3x_j^2} = \frac{(101 - i)^3}{101(3i^2 - 303i + 101^2)}.$$

Hence

$$\frac{x_i^3}{1 - 3x_i + 3x_i^2} + \frac{x_j^3}{1 - 3x_j + 3x_j^2} = 1.$$

So the sum is 51.

2. Given the following triangular arrangements of circles, each of the numbers $1, 2, \ldots, 9$ is to be written into one of these circles, so that each circle contains exactly one of these numbers and



(i) the sums of the four numbers on each side of the triangle are equal;

(ii) the sums of the squares of the four numbers on each side of the triangles are equal.

Find all ways in which this can be done.

Official solution.

Let s be the sum of the fours numbers on each side of the triangle and let S be the sum of the squares of the four numbers on each side of the triangle. Let x, y, z be the numbers in the corners of the triangle, with $x < y < z$. Finally, let a, b, $a < b$, be the two numbers on the same side as y, z.

$$3s = 45 + x + y + z$$
$$3S = 285 + x^2 + y^2 + z^2$$

Thus

$$3 \mid x + y + z \quad \text{and} \quad 3 \mid x^2 + y^2 + z^2$$

and it follows that $x \equiv y \equiv z \pmod 3$.

Case 1: $x = 3, y = 6, z = 9$: In this case $s = 21$ and $S = 137$. Thus

$$a + b = 5 \quad \text{and} \quad a^2 + b^2 = 20.$$

So there is no solution.

Case 2: $x = 1, y = 4, z = 7$. In this case $s = 19$ and $S = 117$. Thus

$$a + b = 8 \quad \text{and} \quad a^2 + b^2 = 52.$$

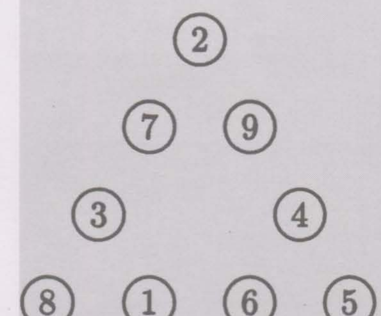
So there is no solution.

Case 3: $x = 2, y = 5, z = 8$. Then $s = 20, S = 126$. In this case $s = 21$ and $S = 137$. Thus

$$a + b = 7 \quad \text{and} \quad a^2 + b^2 = 37.$$

So $a = 1, b = 6$.

By similar considerations, the numbers on the other sides can be found to be unique. The solution is shown in the picture.



Thus there are 48 solutions since there 6 ways to place x, y, z at the corners and for each of these, there are 8 ways to place the remaining 6 numbers.

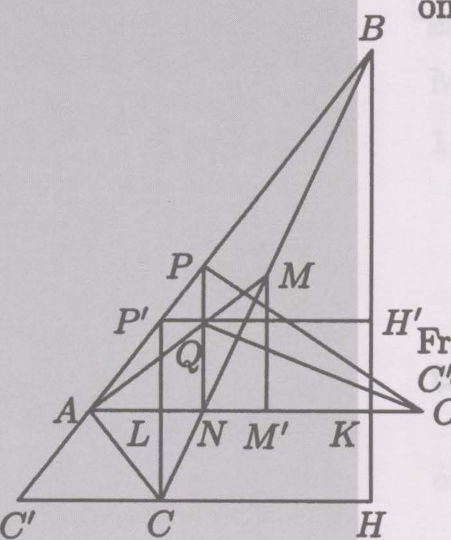
3. Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced.

Prove that QO is perpendicular to BC.

Solutions by R. Pargeter (England) and Lu Shangyi (National University of Singapore). We first present the official solution.

If $\angle B = \angle C$, the proof is obvious. So we suppose without loss of generality that $\angle B < \angle C$. Produce BA to C' so that $AC' = AC$. Then $CC' \| AN$. Let BH be the perpendicular from B onto $C'C$. Draw CP' parallel to HB, intersecting AN in L and AB in P'. Then AN produced bisects HH', at K say, where

$H' \in BH$ and $P'H' \| CC'$. Draw the perpendicular MM' from M onto AN produced.



Since M is the midpoint of BC,

$$M'M = \frac{KB - KH}{2} = \frac{KB - KH'}{2} = \frac{H'B}{2},$$

$$AM' = AL + \frac{LK}{2} = \frac{C'H}{2}.$$

From similar triangles we have $NQ : AN = M'M : AM' = H'B : C'H$ or $NQ : H'B = AN : C'H = NP : HB$. Therefore

$$NQ : NP = H'B : HB = CH : C'H.$$

But

$$NP : NO = AN : NP = C'H : HB.$$

Therefore $NQ : NO = CH : HB$. Hence the right triangles ONQ, CHB are similar, and since ON is perpendicular to HB, OQ must also be perpendicular to BC.

Next is Lu's solution using coordinate geometry. Pargeter also has a solution along this line.

Let us set up a coordinate system. Let N be the origin, with NO as the x-axis and NP as the y-axis. Let the coordinates of P be $(0, c)$ and the gradient of the line AB be $m_{AB} = m$. Then the equation of the line AB is given by $y = mx + c$. Since AN is the angle bisector at A, we have the equation of the line AC to be $y = -mx - c$. Since BC passes through the origin, its equation is of the form $y = ax$. Then the coordinates of A, B and C are

$$A\left(\tfrac{-c}{m}, 0\right), \quad B\left(\tfrac{c}{a-m}, \tfrac{ac}{a-m}\right), \quad C\left(\tfrac{-c}{a+m}, \tfrac{-ac}{a+m}\right).$$

Since PO is perpendicular to AB, $m_{PO} = -1/m$ and the equation of PO is given by $y = -\frac{x}{m} + c$. Thus $O(cm, 0)$. Since M is the midpoint of BC, its coordinates are given by $M(mc/(a^2 - m^2), amc/(a^2 - m^2))$. Now MA intersects the y-axis at Q. Thus $x = 0$ and $y = mc/a$. Hence the coordinates of Q are given by $Q(0, mc/a)$. Hence $m_{OQ} = -1/a$ and $m_{OQ}m_{BC} = -1$ and hence OQ is perpendicular to BC.

4. Let n, k be given positive integers with $n > k$. Prove that

$$\frac{1}{n+1} \cdot \frac{n^n}{k^k(n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k(n-k)^{n-k}}.$$

Solution.

Let $b = n - k$. We are required to prove that

$$n^n > k^k b^b \binom{n}{k} \quad \text{and} \quad n^n < (n+1) k^k b^b \binom{n}{k}.$$

We have

$$n^n = (k+b)^n = \binom{n}{0} k^0 b^n + \binom{n}{1} k^1 b^{n-1} + \cdots + \binom{n}{n} k^n b^0.$$

Since $\binom{n}{k} k^k b^b$ is one of the terms on the right, we have

$$n^n > \binom{n}{k} k^k b^b.$$

Next, for any $j > 0$,

$$\frac{\binom{n}{k} k^k b^b}{\binom{n}{k+j} k^{k+j} b^{b-j}} = \frac{(k+j)!(b-j)!b^j}{k!b!k^j}$$
$$= \frac{(k+j)(k+j-1)\cdots(k+1)}{k^j} \frac{b^j}{b(b-1)\cdots(b-j-1)} > 1$$

$$\frac{\binom{n}{k} k^k b^b}{\binom{n}{k-j} k^{k-j} b^{b+j}} = \frac{(k-j)!(b+j)!k^j}{k!b!b^j}$$
$$= \frac{(b+j)(b+j-1)\cdots(b+1)}{b^j} \frac{k^j}{k(k-1)\cdots(k-j-1)} > 1$$

Since each term on the right is $\geq \binom{n}{k} k^k b^b$, we have

$$n^n > (n+1) \binom{n}{k} k^k b^b.$$

5. Given a permutation $(a_0, a_1, \ldots, a_n)$ of the sequence $0, 1, \ldots, n$. A transposition of a_i with a_j is called *legal* if $i > 0, a_i = 0$ and $a_{i-1} + 1 = a_j$. The permutation $(a_0, a_1, \ldots, a_n)$ is called *regular* if after a number of legal transpositions it becomes $(1, 2, \ldots, n, 0)$. For which numbers n is the permutation $(1, n, n-1, \ldots, 3, 2, 0)$ regular?

Solution.

Let P_n denote the permutation $(1, n, n-1, \ldots, 3, 2, 0)$. First we observe that P_n is trivially regular for $n = 1, 2$. Now consider the case $n \geq 3$.

By a sequence of legal transpositions $a, b, c, \ldots$ we mean 0 is legally transposed with a, then b, then c and so on.

If n is even, then after the sequence of legal transpositions $3, 5, 7, \ldots, n-1$, P_n will be transformed into a permutation where 0 will be on the right of n and no further legal transposition is possible. For example

$$(1,8,7,6,5,4,3,2,0) \quad \text{is transformed into} \quad (1,8,0,6,7,4,5,2,3).$$

Thus n is not regular if n is even.

So we assume that n is odd. We can write $n = k2^j - 1$, where k, j are positive integers. The sequence of legal transpositions $3, 5, \ldots, n$ transforms P_n into

$$Q_n = (1, 0, n-1, n, n-3, n-2, \ldots, 2, 3).$$

We shall encounter permutations like this frequently. So we introduce the notation:

$$\pi(a,b) = [1, 2^b - 1]0[a2^b, (a+1)2^b - 1]$$
$$[(a-1)2^b, a2^b - 1] \ldots [2.2^b, 3.2^b - 1][2^b, 2.2^b - 1]$$

where for any two integers $s \le t$, $[s,t] = (s, s+1, \ldots, t)$. Thus $\pi(a,b)$ is a permutation of $n = (a+1)2^b - 1$. For example:

$$\pi(3,2) = [1,3]0[12,15][8,11][4,7]$$
$$= (1,2,3,0,12,13,14,15,8,9,10,11,4,5,6,7).$$

Also

$$Q_n = \pi((n-1)/2, 1) = \pi(k2^{j-1} - 1, 1).$$

If $a = 2\ell + 1$ is odd, then $\pi(a,b)$ can be transformed, by the legal transpositions

$$2^b, 3.2^b, \ldots, a2^b,$$
$$2^b + 1, 3.2^b + 1, \ldots, a.2^b + 1,$$
$$\ldots\ldots\ldots$$
$$2^b + 2^b - 1, 3.2^b + 2^b - 1, \ldots, a.2^b + 2^b - 1, 2^b + 2^b$$

into

$$[1, 2^b - 1][2^b, 2.2^b - 1]0[(a-1)2^b, a2^b - 1][a2^b, (a+1)2^b - 1]$$
$$\ldots [2.2^b, 3.2^b - 1][3.2^b, 4.2^b - 1]$$
$$= [1, 2^{b+1} - 1]0[\ell 2^{b+1}, (\ell+1)2^{b+1} - 1] \ldots [2^{b+1}, 2.2^{b+1} - 1]$$
$$= \pi(\ell, b+1)$$

Thus $\pi(3,2)$ is tranformed into

$$\pi(1,3) = (1,2,3,4,5,6,7,0,8,9,10,11,12,13,14,15).$$

If a is even, then the legal transpositions $2^b, 3.2^b, \ldots, (a-1)2^b$ transform $\pi(a,b)$ into

$$[1, 2^b - 1]2^b[a2^b, (a+1)2^b - 1]0[(a-1)2^b + 1, a2^b - 1]$$
$$\ldots 5.2^b[3.2^b + 1, 3.2^b - 1][2.2^b, 3.2^b - 1]3.2^b[2^b + 1, 2.2^b - 1].$$

From this no further legal transposition is possible since 0 is now on the right of $n = (a+1)2^b - 1$.

If $n = k2^j - 1$, then P_n can be transformed into $Q_n = \pi(k2^{j-1} - 1, 1)$. From this it can be transformed into $\pi(k2^{j-2} - 1, 2), \ldots, \pi(k-1, j)$.

If $k = 1$, $\pi(0,1) = [1, n]$. If $k > 1$, then $k - 1$ is even, and we know that $\pi(k-1, j)$ cannot be legally transformed to $[1, n]$.

Thus the only n which are regular are those that can be written in the form $n = 2^j - 1$.