Contest

1. Prizes in the form of book vouchers will be awarded to the first received best solution(s) submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

3. Solutions should be typed and sent to:
The Editor,
Mathematics Medley,
c/o Department of Mathematics,
National University of Singapore,
2 Science Drive 2,
Singapore 117543;
and should arrive before
31 August 2001.

4. The Editor's decision will be final and no correspondence will be entertained.

Problems Corner

Problem 1

The numbers 1, 2, 3, ..., 2001 are arranged in a sequence. If the first term is k, then the first k terms of this sequence is rearranged in the reversed order. Is it always possible to obtain 1 as the first term by applying a finite number of this operation to the sequence?

Prize

One $150 book Voucher

Problem 2

Let b and c be positive integers such that b divides $c^2 + 1$ and c divides $b^2 + 1$. Determine the value of

$$\frac{b}{c}+\frac{c}{b}+\frac{1}{bc}.$$

Prize

One $150 book Voucher

SOLUTIONS

Solutions to the problems of volume 27, No.2, 2000

Problem 1

Find all positive integers n such that $n + s(n) = 2001$, where $s(n)$ is the sum of all the digits of n.

PRIZE
one $100 book voucher

Solution I

by *Nguyen Thi Thao Nguyen*,
CHIJ Secondary School

Clearly, $n \leq 2000$. This implies that $s(n) \leq 1 + 9 \times 3 = 28$. Thus, $n = 2001 - s(n) \geq 2001 - 28 = 1973$. Let $n = \overline{19ab}$, where a is the tens digit, b the units digit of n and $0 \leq a, b \leq 9$. We have $\overline{19ab} + 1 + 9 + a + b = 2001$, which is equivalent to $11a + 2b = 91$. From this, we see that a is an odd integer. On the other hand, $0 \leq b \leq 9$ implies that $91 \geq 91 - 2b = 11a \geq 73$. As a is an integer, this implies that $8 \geq a \geq 7$. Consequently, $a = 7$ and $b = 7$. Hence, the answer is $n = 1977$.

Solution II

by *Ong Xing Cong*,
Raffles Institution

Note that $s(n) \equiv n \pmod 3$. By taking mod 3 of the equation $s(n) + n = 2001$, we have $2n \equiv s(n) + n \equiv 0 \pmod 3$. Hence $n \equiv 0 \pmod 3$. If n has less than 4 digits, the maximum value of $s(n) + n$ is $999 + 9 + 9 + 9 = 1026 < 2001$. Therefore n has 4 digits and it is less than 2001. Now $s(n)$ is at most $1 + 9 + 9 + 8 = 27$ and at least 3 as it is positive and divisible by 3. Hence, $s(n)$ can only be $3, 6, 9, 12, 15, 18, 21, 24, 27$. Then $n = 2001 - s(n)$ can only be $1998, 1995, 1992, 1989, 1986, 1983, 1980, 1977, 1974$. Direct checking shows that 1997 is the only answer.

Editor's note: Solved also by Colin Tan Wei Yu, Raffles Institution, Ong Chin Siang, Raffles Institution, Chen Huijing, Victoria Junior College, Serene Lee Min Wai, Raffles Girls' Secondary School, Calvin Lin Zhiwei, Hwa Chong Junior College, Xing Dongfeng, Victoria Junior College. One incorrect solution was received. The prize was shared equally between by Nguyen Thi Thao Nguyen and Ong Xing Cong.

Problem 2

Prove that for any positive real numbers *a, b, c,*

$$\frac{a}{10b+11c}+\frac{b}{10c+11a}+\frac{c}{10a+11b}\geq\frac{1}{7}.$$

PRIZE

one \$100 book voucher

Solution I

by *Calvin Lin Zhiwei*,
Hwa Chong Junior College

Let $x=10a+11b, y=10b+11c, z=10c+11a$. Then $a=(100x-110y+121z)/2331, b=(100y-110z+121x)/2331, c=(100z-110x+121y)/2331$. Thus,

$$\begin{aligned}
&\tfrac{a}{10b+11c}+\tfrac{b}{10c+11a}+\tfrac{c}{10a+11b}\\
=&\ \tfrac{100x-110y+121z}{2331y}+\tfrac{100y-110z+121x}{2331z}+\tfrac{100z-110x+121y}{2331x}\\
=&\ \tfrac{1}{2331}\left[100\left(\tfrac{x}{y}+\tfrac{y}{x}+\tfrac{x}{z}+\tfrac{z}{x}+\tfrac{y}{z}+\tfrac{z}{y}\right)-330+21\left(\tfrac{z}{y}+\tfrac{x}{z}+\tfrac{y}{x}\right)\right]\\
\geq&\ \tfrac{1}{2331}\left[100(2\sqrt{\tfrac{x}{y}\tfrac{y}{x}}+2\sqrt{\tfrac{x}{z}\tfrac{z}{x}}+2\sqrt{\tfrac{y}{z}\tfrac{z}{y}})-330+21\times 3\sqrt[3]{\tfrac{z}{y}\tfrac{x}{z}\tfrac{y}{x}}\right]\\
=&\ \tfrac{1}{2331}\left[100(2+2+2)-330+21\times 3\right]=\tfrac{1}{7}.
\end{aligned}$$

Clearly, equality hold if and only if $x=y=z$ if and only if $a=b=c$.

Solution II

by *Xing Dongfeng*,
Victoria Junior College

By Cauchy-Schwarz inequality, we have

$$\left[\left(\sqrt{\tfrac{a}{10b+11c}}\right)^2+\left(\sqrt{\tfrac{b}{10c+11a}}\right)^2+\left(\sqrt{\tfrac{a}{10a+11b}}\right)^2\right]\left[\begin{matrix}\left(\sqrt{a(10b+11c)}\right)^2+\left(\sqrt{b(10c+11a)}\right)^2\\+\left(\sqrt{c(10a+11b)}\right)^2\end{matrix}\right]$$

$$\geq\left[\sqrt{\tfrac{a}{10b+11c}}\sqrt{a(10b+11c)}+\sqrt{\tfrac{b}{10c+11a}}\sqrt{b(10c+11a)}+\sqrt{\tfrac{c}{10a+11b}}\sqrt{a(10a+11b)}\right]^2.$$

That is $\left(\dfrac{a}{10b+11c}+\dfrac{b}{10c+11a}+\dfrac{c}{10a+11b}\right)(21ab+21bc+21ca)\geq(a+b+c)^2.$

Since $(a+b+c)^2=a^2+b^2+c^2+2(ab+bc+ca)\geq(ab+bc+ca)+2(ab+bc+ca)=3(ab+bc+ca)$, we have

$$\frac{a}{10b+11c}+\frac{b}{10c+11a}+\frac{c}{10a+11b}\geq\frac{(a+b+c)^2}{21(ab+bc+ca)}\geq\frac{3(ab+bc+ca)}{21(ab+bc+ca)}=\frac{1}{7}.$$

Editor's note: Solved also by Colin Tan Wei Yu, Raffles Institution, Wu Zhenyu, Victoria Junior College. The prize was shared equally between by Calvin Lin Zhiwei and Xing Dongfeng.