COUNTING – Its Principles & Techniques (11)

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27. The Stirling Numbers of the Second Kind

In Section 25 (see [3]), we introduced the Stirling number of the first kind $s(m,k)$ that is defined as the coefficient of x^k in the expansion of

$$[x]_m = x(x-1)\cdots(x-m+1);$$

namely,

$$[x]_m = \sum_{i=0}^{m} s(m,i)\, x^i . \tag{27.1}$$

The sequence of numbers $s(m,1), s(m,2), \cdots, s(m,m)$ alternate in sign with $s(m,1)$ positive when and only when m is odd.

In Section 26 (see also [3]), we gave a combinatorial interpretation of $s(m,k)$; that is, the absolute value of $s(m,k)$ is the number of ways of arranging m distinct objects around k identical circles with at least one object at each circle.

In this section, we shall introduce the other sequence of Stirling numbers, called the *Stirling numbers* of the *second kind.*

Let us begin with a simple example. Consider 4 distinct objects: a, b, c and d. Clearly, there is one and only one way to group them into '1' group, that is, $\{a,b,c,d\}$; and there is one and only one way to divide them into 'four' groups, that is,

$$\{a\}\cup\{b\}\cup\{c\}\cup\{d\}.$$

Now, (i) how many ways are there to divide them into 'two' groups?

There are 7 ways as shown below:

$$\begin{aligned}&\{a,b,c\}\cup\{d\},\ \{a,b,d\}\cup\{c\},\ \{a,c,d\}\cup\{b\},\\ &\{b,c,d\}\cup\{a\},\ \{a,b\}\cup\{c,d\},\ \{a,c\}\cup\{b,d\},\\ &\{a,d\}\cup\{b,c\}.\end{aligned}$$

(ii) How many ways are there to divide them into 'three' groups?

There are 6 ways as shown below:

$$\begin{aligned}&\{a,b\}\cup\{c\}\cup\{d\},\ \{a,c\}\cup\{b\}\cup\{d\},\ \{a,d\}\cup\{b\}\cup\{c\},\\ &\{b,c\}\cup\{a\}\cup\{d\},\ \{b,d\}\cup\{a\}\cup\{c\},\ \{c,d\}\cup\{a\}\cup\{b\}.\end{aligned}$$

Given two positive integers *n* and *k* with $k \le n$, the *Stirling number* of the *second kind*, denoted by $S(n,k)$, is defined as the number of ways of dividing *n distinct* objects into *k* (nonempty) groups; that is, the number of ways of partitioning an *n*-element set into *k* nonempty subsets. Thus, as shown in the above example, we have

$$S(4,1)=1,\ S(4,2)=7,\ S(4,3)=6,\ S(4,4)=1.$$

Problem 27.1. Find the value of $S(5,k)$, where $k=1,2,3,4,5$.

Example 27.1. Find the number of ways to express 2730 as a product *ab* of two numbers *a* and *b*, where $a>b\ge 2$.

Observe that $2730=2\cdot 3\cdot 5\cdot 7\cdot 13$, and such a pair *a*, *b* of factors is obtained by dividing $\{2,3,5,7,13\}$ into 2 groups (and then taking their products). Thus, the desired number of ways is given by $S(5,2)(=15)$. □

Problem 27.2. Find, in terms of $S(n,k)$, the number of ways to express 39270 as a product *abc* of three integers *a*, *b* and *c*, where $a>b>c\ge 2$.

Problem 27.3. It is clear that

(i) $S(n,1)=S(n,n)=1$,

(ii) $S(n,k)=0$ if $k>n\ge 1$, and

(iii) $S(n,0)=S(0,k)=0$ if $n\ge 1$ and $k\ge 1$.

We define

(iv) $S(0,0)=1$.

Show that for $n\ge 1$,

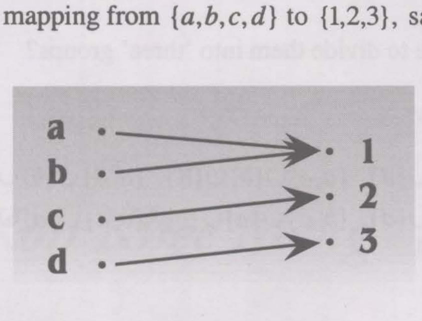
(v) $S(n,2)=2^{n-1}-1$,

(vi) $S(n,n-1)=\binom{n}{2}$.

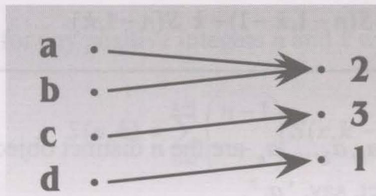
28. The Number of Onto Mappings

We pointed out in Section 19 (see [1]) that the problem of counting the number of onto mappings from a finite set to another finite set is not straight forward, and we showed by an example how to tackle this problem by applying (PIE). In this section, we shall point out that this counting problem is actually closely related to the problem of evaluating the $S(n,k)$'s.

Consider an onto mapping from $\{a,b,c,d\}$ to $\{1,2,3\}$, say,



This onto mapping can be regarded as first dividing the 4 elements *a*, *b*, *c*, *d* into 3 groups: $\{a,b\}$, $\{c\}$, $\{d\}$, and then naming the groups as '1', '2' and '3' respectively. If we rename the groups as '2', '3' and '1' respectively, then we get another onto mapping:



Since there are 3! ways to name the 3 groups, we see that a way of dividing 4 distinct objects into 3 groups gives rise to 3! onto mappings from $\{a,b,c,d\}$ to $\{1,2,3\}$. It thus follows that the number of onto mappings from $\{a,b,c,d\}$ to $\{1,2,3\}$ is given by $3!\,S(4,3)(=36)$.

In general, we have:

> The number of *onto* mappings from an *n*-element set to a *k*-element set is given by
>
> $$k!S(n,k). \qquad (28.1)$$

Problem 28.1. In Example 19.1 (see [1]), we applied (PIE) to compute the number of onto mappings from a 5-element set to a 3-element set, which is '150'. Verify this result by applying (28.1) and your answer of $S(5,3)$ in Problem 27.1.

Using the general statement of (PIE) as shown in Section 20 (see [2]), one can show that (see Problem 22.3 in [2]) the number of *onto* mappings from an *n*-element set to a *k*-element set is given by

$$\sum_{r=0}^{k} (-1)^r \binom{k}{r} (k-r)^n .$$

Combining this with (28.1), we have:

> $$S(n,k)=\frac{1}{k!}\sum_{r=0}^{k} (-1)^r \binom{k}{r} (k-r)^n . \qquad (28.2)$$

29. A Recurrence Relation

The formula (28.2) provides us with a way to evaluate $S(n,k)$'s. There is another way to do so. As shown in Section 25 (see [3]), the Stirling numbers of the first kind $s(m,k)$'s satisfy the following recurrence relation:

$$s(m,k) = s(m-1,k-1)-(m-1)\,s(m-1,k).$$

For the Stirling numbers of the second kind, likewise, we have the following recurrence relation:

> For positive integers n, k with $n \geq k$,
>
> $$S(n,k) = S(n-1,k-1) + k\ S(n-1,k). \qquad (29.1)$$

To see why (29.1) holds, suppose $a_1, a_2, ..., a_n$ are the n distinct objects which are divided into k groups. Consider a particular object, say, 'a_1'.

Case 1. 'a_1' itself forms a group.

In this case, the $n-1$ objects $a_2, ..., a_n$ are then divided into $k-1$ groups. By definition, there are $S(n-1,k-1)$ ways of grouping.

Case 2. a_1 is in a group with at least 1 other object.

In this case, the $n-1$ distinct objects $a_2, ..., a_n$ are divided into k groups and there are $S(n-1,k)$ ways of grouping. In any such grouping, a_1 has k choices to be in one of the k groups. Thus, there are $k\ S(n-1,k)$ ways in this case.

The relation (29.1) now follows by (AP). □

Using the initial values shown in Problem 27.3 and applying (29.1), one can find out the values of other $S(n,k)$'s. For instance,

$$S(3,2) = S(2,1) + 2S(2,2) = 1 + 2 \cdot 1 = 3;$$
$$S(4,2) = S(3,1) + 2S(3,2) = 1 + 2 \cdot 3 = 7;$$
$$S(4,3) = S(3,2) + 3S(3,3) = 3 + 3 \cdot 1 = 6;$$

etc.

It is in this way that one can easily construct the following table of the values of $S(n,k)$'s.

r \ n	0	1	2	3	4	5	6	7	8	9
0	1									
1	0	1								
2	0	1	1							
3	0	1	3	1						
4	0	1	7	6	1					
5	0	1	15	25	10	1				
6	0	1	31	90	65	15	1			
7	0	1	63	301	350	140	21	1		
8	0	1	127	966	1701	1050	266	28	1	
9	0	1	255	3025	7770	6951	2646	462	36	1

Table 29.1. The values of $S(n,k)$'s

Problem 29.1. Show that for any positive integers n and k with $n \geq k$,

$$S(n,k) = \sum_{r=0}^{n-1} \binom{n-1}{r} S(r,k-1).$$

30. Expressing x^n in terms of $[x]_i$'s

As shown in (27.1), when $[x]_m$ is expressed in terms of x^i's, the Stirling numbers of the first kind are the coefficients. Suppose, conversely, we wish to express x^n in terms of $[x]_i$'s. What can be said about the coefficients? To answer this question, let us consider the following counting problem:

Let $N_n = \{1,2,\cdots,n\}$. Determine α, the number of mappings from N_n to N_k.

We shall now use two different methods to count α. The first method is the 'natural' one:

$$\alpha = \underbrace{k\cdots k}_{n} = k^n. \tag{30.1}$$

The second method is a 'stupid' one. According to the size $|f(N_n)|$ of the image of a mapping $f: N_n \to N_k$, the set of mappings from N_n to N_k can be partitioned into k groups A_i, $i = 1,\cdots,k$, where A_i consists of those mappings whose images have exactly i elements, i.e.,

$$A_i = \{f : N_n \to N_k \,\|\, f(N_n)| = i\}.$$

What is the value of $|A_i|$? Well, $|A_i|$ counts the number of onto mappings from N_n to an i-element subset of N_k. There are $\binom{k}{i}$ ways to choose an i-element subset of N_k, and the number of onto mappings from N_n to this chosen i-element subset is $i!S(n,i)$ by (28.1). Thus

$$\begin{aligned}|A_i| &= \binom{k}{i} i!S(n,i)\\ &= k(k-1)\cdots(k-i+1)\, S(n,i)\\ &= [k]_i\, S(n,i).\end{aligned}$$

Now, by (AP), we have

$$\begin{aligned}\alpha &= \sum_{i=1}^{k} |A_i| \\ &= \sum_{i=1}^{k} [k]_i \; S(n,i) \\ &= \sum_{i=1}^{n} [k]_i \; S(n,i) \qquad ([k]_i = 0 \text{ if } i \geq k+1).\end{aligned}$$

Comparing this result with (30.1) both count for α, we have:

$$k^n = \sum_{i=1}^{n} [k]_i \; S(n,i).$$

If we replace '*k*' by a real variable '*x*', we then obtain:

$$x^n = \sum_{i=1}^{n} S(n,i)[x]_i. \qquad (30.2)$$

Thus we see that when x^n is expressed in terms of $[x]_i$'s, the Stirling numbers of the second kind are the coefficients.

For instance, when $n = 4$,

$$\begin{aligned}\sum_{i=1}^{4} S(4,i)[x]_i &= S(4,1)[x]_1 + S(4,2)[x]_2 + S(4,3)[x]_3 + S(4,4)[x]_4 \\ &= 1x + 7x(x-1) + 6x(x-1)(x-2) + 1x(x-1)(x-2)(x-3) \\ &= x + 7x^2 - 7x + 6x^3 - 18x^2 + 12x + x^4 - 6x^3 + 11x^2 - 6x \\ &= x^4.\end{aligned}$$

Answers to problems

Problem 27.1. $S(5,1) = 1,\ S(5,2) = 15,\ S(5,3) = 25,\ S(5,4) = 10,\ S(5,5) = 1.$

Problem 27.2. $S(6,3)$. (Observe that $39270 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 17$)

References.

[1] K. M. Koh, *Counting-Its Principles and Techniques (7)*, Mathematical Medley Vol. 25 March (1999) 63 – 74.

[2] K. M. Koh, *Counting-Its Principles and Techniques (8)*, Mathematical Medley Vol. 26 July (1999) 4 – 8.

[3] K. M. Koh, *Counting-Its Principles and Techniques (10)*, Mathematical Medley Vol. 27 Dec (2000) 60 – 68.