COMPETITION CORNER

In this issue we publish the problems of the First Hong Kong (China) Mathematical Olympiad Contest 1999, Greek National Mathematical Olympiad 2000, and XII Asian Pacific Mathematical Olympiad, March 2000.

Please send your solutions of these Olympiads to the address given above. All correct solutions will be acknowledged. We also present solutions of the 12th Nordic Mathematical Contest 1998, the 1st Japan Mathematical Olympiad 1991, the Georgian Mathematical Olympiad 1997 and the 40th International Mathematical Olympiad 1999.

All communications about this column should be sent to :

Dr. Tay Tiong Seng,
Department of Mathematics,
National University of Singapore,
2 Science Drive 2,
Singapore 117543
or by e-mail to mattayts@nus.edu.sg.

• Problems •

First Hong Kong (China) Mathematical Olympiad

1999

1. $PQRS$ is a cyclic quadrilateral with $\angle PSR = 90°$; H, K are the feet of the perpendiculars from Q to PR, PS (suitably extended if necessary), respectively. Show that HK bisects QS.

2. The base of a pyramid is a convex polygon with 9 sides. Each of the diagonals of the base and each of the edges on the lateral surface of the pyramid is coloured either black or white. Both colours are used. (Note that the sides of the base are not coloured.) Prove that there are three segments coloured the same colour which form a triangle.

3. Let s, t be given nonzero integers, and let (x, y) be any ordered pair of integers. A move changes (x, y) to $(x + t, y - s)$. The pair (x, y) is **good** if after some (may be zero) number of moves it describes a pair of integers that are **not** relatively prime.

(a) Determine if (s, t) is a good pair.

(b) Show that for any s and t there is pair (x, y) which is not good.

4. Let f be a function defined on the positive reals with the following properties:

(1) $f(1) = 1$,

(2) $f(x + 1) = xf(x)$,

(3) $f(x) = 10^{g(x)}$, where $g(x)$ is a function defined on the reals satisfying

$$g(ty + (1 - t)z)) \leq tg(y) + (1 - t)g(z)$$

for all y and z and for $0 \leq t \leq 1$.

(a) Prove that

$$t[g(n) - g(n - 1)] \leq g(n + t) - g(n) \leq t[g(n + 1) - g(n)]$$

where n is an integer and $0 \leq t \leq 1$.

(b) Prove that $\frac{4}{3} \leq f(\frac{1}{2}) \leq \frac{4}{3}\sqrt{2}$.

Greek National Mathematical Olympiad 2000

1. Consider the rectangle $ABCD$ with $AB = \alpha$, $AD = \beta$. A line ℓ passing through the centre O of the rectangle meets the side AD at the point E such that $AE/ED = 1/2$. On this line take an arbitrary point M lying inside the rectangle. Find the necessary and sufficient condition on α and β so that distances from M to the sides of the rectangle AD, AB, DC and BC, taken in that order, form an arithmetic progression.

2. Find the prime number p so that $1 + p^2 + p^3 + p^4$ is a perfect square, i.e. the square of an integer.

3. Find the maximum positive real number k such that

$$\frac{xy}{\sqrt{(x^2 + y^2)(3x^2 + y^2)}} \leq \frac{1}{k}$$

for all positive real numbers x and y.

4. For the subset $A_1, \ldots, A_{2000}$ of the set M, we have $|A_i| \geq 2|M|/3$, $i = 1, 2, \ldots, 2000$, where $|X|$ denotes the cardinality of the set X. Prove that there exists $\alpha \in M$ which belongs to at least 1334 from the subsets A_i.

XII Asian Pacific Mathematical Olympiad

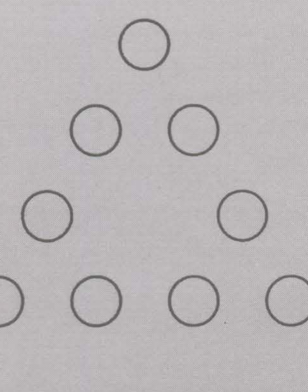
March 2000

1. Compute the sum

$$S = \sum_{i=0}^{101} \frac{x_i^3}{1 - 3x_i + 3x_i^2}$$

for $x_i = \frac{i}{101}$.

2. Given the following triangular arrangements of circles:



Each of the numbers $1, 2, \ldots, 9$ is to be written into one of these circles, so that each circle contains exactly one of these numbers and

(i) the sums of the four numbers on each side of the triangle are equal;

(ii) the sums of the squares of the four numbers on each side of the triangles are equal.

Find all ways in which this can be done.

3. Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively, and O the point in which the perpendicular at P to BA meets AN produced.

Prove that QO is perpendicular to BC.

4. Let n, k be given positive integers with $n > k$. Prove that

$$\frac{1}{n+1} \cdot \frac{n^n}{k^k(n-k)^{n-k}} < \frac{n!}{k!(n-k)!} < \frac{n^n}{k^k(n-k)^{n-k}}.$$

5. Given a permutation $(a_0, a_1, \ldots, a_n)$ of the sequence $0, 1, \ldots, n$. A transposition of a_i with a_j is called *legal* if $i > 0$, $a_i = 0$ and $a_{i-1} + 1 = a_j$. The permutation $(a_0, a_1, \ldots, a_n)$ is called *regular* if after a number of legal transpositions it becomes $(1, 2, \ldots, n, 0)$. For which numbers n is the permutation $(1, n, n-1, \ldots, 3, 2, 0)$ regular?

• Solutions •

South African Mathematical Olympiad, 1999

Third round

1. How many non-congruent triangles with integer sides and perimeter 1999 can be constructed?

We present the solution by Calvin Lin Zhiwei (Hwachong Junior College). Also solved by Lim Chong Jie (Singapore).

Let a, b, c be the lengths of the sides of the triangle with a the smallest among the three. We are looking for the solutions in positive integers solutions of the following:

$$a + b + c = 1999, \quad a + b > c, \quad a + c > b.$$

Thus $3a \leq a + b + c = 1999$, and we have $a \leq 666$. Also the two inequalities imply that $b, c \leq 999$.

Thus when $1 \leq a \leq 500$, we have the following solutions:

$$(a, b, c) = (a, 1000 - a + k, 999 - k), \quad 0 \leq k \leq a - 1.$$

For $501 \le a \le 666$, we have the following solutions:

$$(a, b, c) = (501 + m, 501 + m + k, 1999 - 1002 - k),$$

where $0 \le m \le 666$, and $0 \le k \le 496 - 3m$ since $1999 - 1002 - k \ge 501 + m$. Thus the total number of solutions is:

$$1 + 2 + \cdots + 500 + 497 + 494 + \cdots + 2 = 166667.$$

Since each scalene triangle contributes two solutions and each isosceles triangle contributes one solution and isosceles triangles occur only when a is odd, the number of noncongruent triangles is

$$\frac{166667 + 333}{2} = 83500.$$

2. A, B, C and D are points on a given straight line, in that order. Construct a square $PQRS$, with all of P, Q, R and S on the same side of AD, such that A, B, C and D lie on PQ, SR, QR and PS produced, respectively.

We present the solution by Nicholas Tham Ming Qiang (Raffles Junior College). Also solved by Gary Yeh Yuan Long (Anglo-Chinese Junior College).

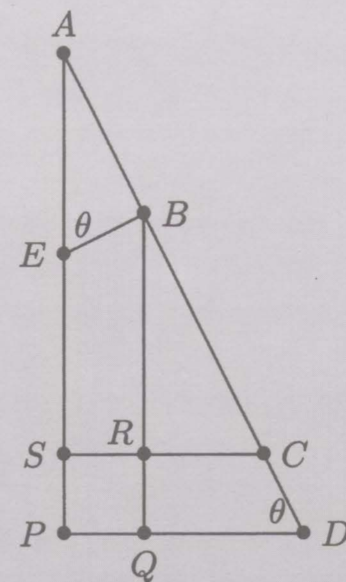
Let $\angle ADP = \theta$. Then $SR = AB\cos\theta$ and $QR = CD\sin\theta$. Thus $\tan\theta = AB/CD$. Thus the construction can be done as follows:

1. Construct the segment BE perpendicular to AB such that $BE = CD$. ($\angle AEB = \theta$.)

2. Construct the line l parallel to AE passing through B.

3. Construct the lines m, n perpendicular to AE and passing through C, D, respectively.

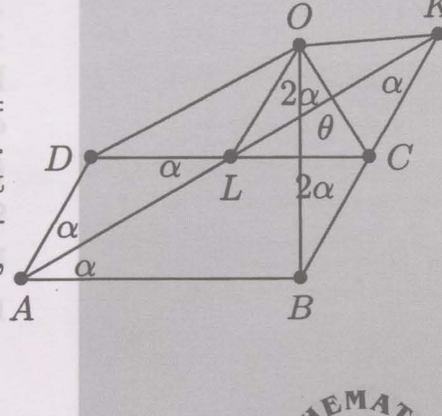
4. The pairwise intersections of the lines AE, l, m, n give the vertices of the square.



3. The bisectors of angle BAD in the parallelogram $ABCD$ intersects the lines BC and CD at the point K and L, respectively. Prove that the centre of the circle passing through the points C, K and L lies on the circle passing through the points B, C and D.

We present the solution by A. Robert Pargeter (England).

Let O be the centre of the circumcircle of LCK (so that $OL = OC = OK$). Denote $\angle DAB$ by 2α. Then $\angle LKC = \angle DAL = \alpha$. Therefore $\angle LOC = 2\angle LKC = 2\alpha$, $\angle DLA = \angle LAB = \alpha$. Let $\angle OCL = \theta$. Then $\angle DLO = 2\alpha + \theta = \angle OCB$. Thus triangles DLO and BCO are congruent. Therefore $\angle ODC = \angle OBC$, whence $ODBC$ is cyclic.



4. The sequence $L_1, L_2, L_3, \ldots$ is defined by

$$L_1 = 1,\ L_2 = 3,\ L_n = L_{n-1} + L_{n-2} \text{ for } n > 2,$$

so the first six terms are $1, 3, 4, 7, 11, 18$. Prove that $L_p - 1$ is divisible by p if p is prime.

We present the solution by Charmaine Sia Jia Min(Raffles Girls' School). Also solved by Ernest Chong (Raffles Institution).

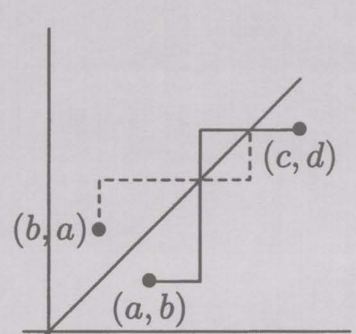
First we note that the result holds for $p = 2$. Henceforth, we assume that $p \geq 3$. The characteristic equation for the recurrence is $x^2 - x - 1 = 0$ whose solution is $a = (1+\sqrt{5})/2$ and $b = (1-\sqrt{5})/2$. Thus $L_n = Aa^n + Bb^n$. By considering $L_1 = 1, L_2 = 3$, we get $A = B = 1$. Thus

$$\begin{aligned} L_p &= \left(\frac{1+\sqrt{5}}{2}\right)^p + \left(\frac{1-\sqrt{5}}{2}\right)^p \\ \Rightarrow \quad 2^{p-1} L_p &= 1 + \binom{p}{2} 5 + \binom{p}{4} 5^2 + \cdots + \binom{p}{p-1} 5^{(p-1)/2} \\ &\equiv 1 \pmod{p} \end{aligned}$$

since $\binom{p}{i} \equiv 0 \pmod{p}$ for $1 \leq i \leq p-1$. By Fermat's Little Theorem, $2^{p-1} \equiv 1 \pmod{p}$ for $p \geq 3$. Thus $L_p \equiv 1 \pmod{p}$ as required.

6. You are at a point (a, b) and need to reach another point (c, d). Both points are below the line $x = y$ and have integer coordinates. You can move in steps of length 1, either upwards or to the right, but you may not move to a point on the line $x = y$. How many different paths are there?

We present the solution by Calvin Lin Zhiwei (Hwachong Junior College).



Clearly, $a \leq c$, and $b \leq d$, otherwise no paths exist.

We now consider two cases. If $a < d$, then there is no restriction and the number of paths is $\binom{c-a+d-b}{c-a}$.

Now suppose $a \geq d$. A path is *good* if it does not touch the line $x = y$. Otherwise it is *bad*. The total number of paths, good or bad, is $\binom{c-a+d-b}{c-a}$. Reflect the rectangle grid with corners at (a, b), (c, d) in the line $x = y$. There is a one to one correspondence between bad paths and paths from (b, a) to (c, d). The total number of such paths is $\binom{c-b+d-a}{c-b}$. Thus the total number of good paths is $\binom{c-a+d-b}{c-a} - \binom{c-a+d-b}{c-b}$. (Note that the second term is 0 if $a < d$. Thus the two cases can actually be combined. In fact under this condition, there is no path from (b, a) to (c, d).)

Austrian-Polish Mathematics Competition

1998

1. Let x_1, x_2, y_1, y_2 be real numbers such that $x_1^2 + x_2^2 \leq 1$. Prove the inequality

$$(x_1y_1 + x_2y_2 - 1)^2 \geq (x_1^2 + x_2^2 - 1)(y_1^2 + y_2^2 - 1).$$

We present the solution by Nicholas Tham Ming Qiang (Raffles Junior College) and Gary Yeh Yuan Long (Anglo-Chinese Junior College).

If $y_1^2 + y_2^2 > 1$, then the result is obvious as the righthand side is negative. So consider the case where $y_1^2 + y_2^2 \leq 1$. Let $x_1 = k \sin\alpha$, $x_2 = k \cos\alpha$, $y_1 = j \sin\beta$, and $y_2 = j \cos\beta$ where $k, j > 0$. Then it suffices to prove that

$$(1 - kj)^2 \geq (1 - k^2)(1 - j^2).$$

This is equivalent to

$$k^2 + j^2 - 2kj \geq 0$$

which is obviously true.

2. Consider n points $P_1, P_2, \ldots, P_n$ lying in that order on a straight line. We colour each point in white, red, green, blue or violet. A colouring is admissible if for each two consecutive points $P_i, P_{i+1} (i = 1, 2, \ldots, n-1)$ either both points are the same colour, or at least one of them is white. How many admissible colourings are there?

Solution by Joel Tay Wei En (Anglo-Chinese School (Independent)).

Let the number of colourings be C_n. Consider the first point. If it is white then there are C_{n-1} admissible colourings. If it is not white then it could be one of 4 colours. Now consider the second point. It is either white, in which case there are C_{n-2} colourings, or the same colour as the first point. Now consider the third point etc., we end up with the recurrence

$$C_n = C_{n-1} + 4(C_2 + C_3 + C_4 + \cdots + C_{n-2}).$$

Replace n by $n-1$ we have

$$C_{n-1} = C_{n-2} + 4(C_2 + C_3 + C_4 + \cdots + C_{n-3}).$$

From these we get the recurrence

$$C_n = 2C_{n-1} + 3C_{n-2}, \quad C_1 = 1, C_2 = 13.$$

The characteristic equation is $x^2 - 2x - 3 = 0$, with distinct roots $x = 3, -1$. Thus $C_n = p3^n + q(-1)^n$. But $C_1 = 3p - q = 1$ and $C_2 = 9p + q = 13$. Solving for p and q, we get $p = 7/6, q = 5/2$. Therefore the solution is $C_n = (7/6)(3^n) + (5/2)(-1^n)$.

3. Find all pairs of real numbers (x, y) satisfying the equations

$$2 - x^3 = y, \qquad 2 - y^3 = x.$$

We present the solution by Calvin Lin Zhiwei (Hwachong Junior College).

The graphs of the two equations are symmetric about the line $x = y$. Thus they intersect only on the line $x = y$. Since $2 - x^3 = x$ implies $(x-1)(x^2 + x + 2) = 0$. But the second factor is always positive. Thus $x = y = 1$ is the only solution.

4. Let m, n be positive integers. Prove that

$$\sum_{k=1}^{n} \lfloor \sqrt[k^2]{k^m} \rfloor \leq n + m(2^{m/4} - 1).$$

We present the solution by Nicholas Tham Ming Qiang (Raffles Junior College). Also solved by Gary Yeh Yuan Long (Anglo-Chinese Junior College).

First we prove that for $k \geq 2$, $\lfloor k^{m/k^2} \rfloor \geq \lfloor (k+1)^{m/(k+1)^2} \rfloor$. This follows from the fact that if $f(x) = x^{1/x}$, then $f'(x) = f(x)(1 - \ln x)/x^2 < 0$ if $x \geq 3$.

When $k > m$, we have

$$k < 2^k \Longrightarrow k < 2^{(k/m)k} \Longrightarrow k^m < 2^{k^2} \Longrightarrow \sqrt[k^2]{k^m} < 2.$$

Thus $\lfloor \sqrt[k^2]{k^m} \rfloor = 1$. So there are at most $m-1$ terms in the summand which is greater than 1. Let i be the number of such terms the largest of which is $b = \lfloor \sqrt[4]{2^m} \rfloor$. Then

$$\sum_{k=1}^{n} \lfloor \sqrt[k^2]{k^m} \rfloor \leq (n - i) + ib \leq n + m(2^{m/4} - 1).$$

5. Find all pairs (a, b) of positive integers such that the equation

$$x^3 - 17x^2 + ax - b^2 = 0$$

has three integer roots (not necessarily distinct).

We present the combined solution by Calvin Lin Zhiwei (Hwa-chong Junior College) and Lim Chong Jie (Singapore).

Let α, β, γ be the three integer roots. Since the left hand side of the equation is negative for $x \leq 0$, we conclude α, β, γ are all positive. We have

$$\alpha\beta\gamma = b^2, \quad \alpha\beta + \beta\gamma + \alpha\gamma = a, \quad \alpha + \beta + \gamma = 17.$$

The following are the solutions to the first and third equation (here we assume without loss of generality that $\alpha \leq \beta \leq \gamma$):

$$(\alpha, \beta, \gamma) = (1, 8, 8), (2, 5, 10), (3, 6, 8), (4, 4, 9).$$

From these we have

$$(a, b) = (80, 8), (80, 10), (88, 12), (90, 12).$$

6. Distinct points A, B, C, D, E, F lie on a circle in that order. The tangents to the circle at the points A and D, and the lines BF and CE are concurrent. Prove that the lines AD, BC, EF are either parallel or concurrent.

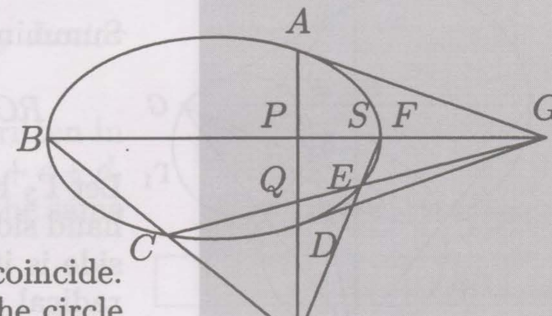
We present the solution by A. Robert Pargeter (England).

This is a very simple exercise in projective geometry–sadly little taught and studied nowadays!

Let BF, CE, etc, meet at G. Let BG, CG meet AD at P, Q, respectively. Let BC meet AD at R. (If $BC||AD$ then R is the point at infinity on AD). Let RE meet BG at S.

Since AD is the polar of G, $BPFG$ and $CQEG$ are harmonic ranges: in brief notation

$$\{BPFG\} = \{CQEG\} = -1.$$



Projecting from R,

$$\{BPSG\} = \{CQEG\} = -1.$$

Therefore $\{BPFG\} = \{BPSG\}$ and hence F and S coincide. Since the proof is strictly projective (i.e., nonmetrical), the circle can replaced by any nondegenerate conic (as in my diagram).

For those not familiar with projective geometry, we present the following solutions.

First we note that when FE is parallel to BC, then they are parallel to AD and the result holds. Thus we assume that BC is not parallel to FE.

Let BF, CE meet at G. Let BG, CG meet AD at P, Q, respectively. We'll prove that $QE/QC = GE/GC$ and $PF/PB = GF/GB$. (Note: This is equivalent to

$$\{CDEG\} = -1 \quad \text{and} \quad \{BDFG\} = -1$$

in Pargeter's solution.)

Since $\triangle GAC \simeq \triangle GEA$, we have $GE/EA = GA/AC$. Also $\triangle GDC \simeq \triangle GED$. Thus $DC/GC = ED/GD$. These imply that $GE/GC = EA \cdot ED/CA \cdot CD$.

By considering $\triangle QAC \simeq \triangle QED$ and $\triangle QCD \simeq QAE$, we have $QS/QC = EA \cdot ED/CA \cdot CD$ and $QE/QC = GE/GC$ as desired. Similarly, we have $PF/PB = GF/GB$.

Let BC meet FE at R and BE meet CF at X and RX meet CE at Q' and BF at P'. We'll prove that $Q'E/Q'C = GE/GC$ and $P'F/P'B = GF/GB$. These will prove that $P' = P$ and $Q' = Q$ and the result will follow.

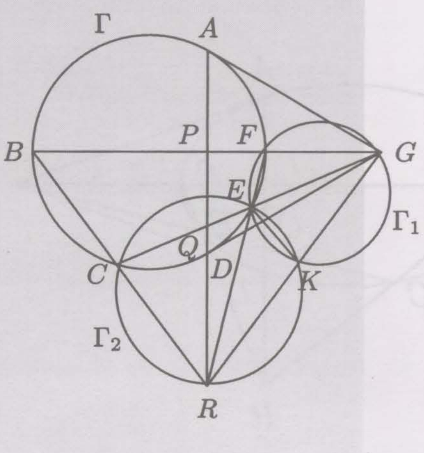
Apply both Ceva's and Menelaus' Theorem to $\triangle CER$ and we'll obtain the first equality. Do the same to $\triangle BFR$ and we'll get the second equality.

Third solution: Let Γ denote the circle. Further let BF and CE meet at G and BC meet FE at R. Let K be the point on the line RG such that G, E, F, K are concyclic. (Note that K is on the segment RG.) Call the circle Γ_1. Then C, E, K, R are concyclic. Call this circle Γ_2. Since R is on the radical axis of Γ and Γ_1, we have

$$RC \cdot RB = RK \cdot RG.$$

Since G is on the radical axis of Γ and Γ_2, we have

$$GD^2 = GK \cdot GR.$$



Summing up we have

$$RC \cdot RB + GD^2 = GR^2, \textit{ i.e.}, \ RC \cdot RB = GR^2 - GD^2.$$

Let Γ_3 be the circle with centre G and radius GD. Then the left hand side is the power of R with respect to Γ while the right hand side is its power with respect to Γ_3. Therefore R is on AD, the radical axis of the two circles.

7. Consider all pairs (a, b) of natural numbers such that the product $a^a b^b$, written in base 10, ends with exactly 98 zeroes. Find the pair (a, b) for which the product ab is smallest.

We present the solution by Nicholas Tham Ming Qiang (Raffles Junior College).

Let $(x)_i$ denote the largest integer k such that $i^k \mid x$. Thus

$$\min\{(a^a b^b)_2, (a^a b^b)_5\} = 98. \qquad (*)$$

Now suppose $5 \mid a$, and $5 \mid b$. Then $(a^a b^b)_5 = a(a)_5 + b(b)_5$. Thus $5 \mid (a^a b^b)_5$. Similarly $5 \mid (a^a b^b)_2$. This violates $(*)$ and so this case is impossible. Thus we assume without loss of generality that $5 \nmid a$ and $5 \mid b$. (Note that $5 \mid ab$.)

Suppose $(b)_5 = 1$. Since $(a^a b^b)_5 = (b^b)_5 \geq 98$, we have $b > 98$. If $2 \mid b$, then we have $(a^a b^b)_2 \geq (b^b)_2 > 98$. This again violates $(*)$. Thus in this case, b must be odd the minimum such b is 105. Also $(a^a b^b)_2 = (a^a)_2 = 98$. This implies that $a = 98$. Thus $(a, b) = (98, 105)$.

Suppose $(b)_5 = 2$. Then $(a^a b^b)_5 = (b^b)_5 = 2b \geq 98$. Thus $b = 50, 75$ or > 98. Consider the case $b > 98$. If $(b)_2 > 0$, then $(a^a b^b)_2 \geq (b^b)_2 > b > 98$ which violates $(*)$. If $(b)_2 = 0$, then $(a^a b^b)_2 = a(a)_2 = 98$. This implies $a = 98$ and yields the minimum solution $(a, b) = (98, 125)$. But this is larger than the previous solution and is discarded.

For $b = 50$, there is no solution. For $b = 75$, we get $a = 98$. This is the best solution so far.

The final case is $(b)_5 > 2$. For this we get $b > 125$ and $a = 98$. This is again too large. Thus the solution is $(a, b) = (98, 75)$.

8. Let $n > 2$ be a given natural number. In each unit square of an infinite grid is written a natural number. A polygon is admissible if it has area n and its sides lie on the grid lines. The sum of the numbers written in the squares contained in an admissible polygon is called the value of the polygon. Prove that if the values of any two congruent admissible polygons are equal, then all of the numbers written in the squares of the grid are equal.

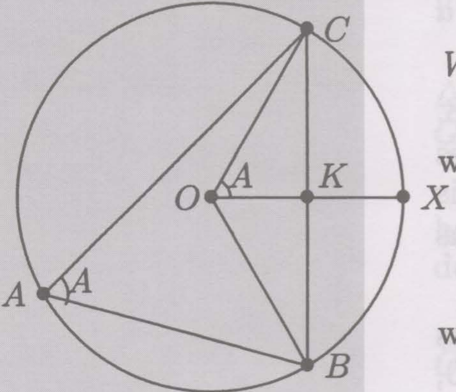
We present the solution by Calvin Lin Zhiwei (Hwachong Junior College). and Lim Chong Jie (Singapore).

Consider any 2×2 grid with the numbers a, b, c, d written in its unit squares (see figure). Then we have $a + c + d = b + c + d$. Thus $a = b$. Hence every two adjacent unit squares carry the same number. Thus all the squares carry the same number.

(Consider any horizontal strip of $n - 1$ unit squares. Since any unit adjacent to this strip will form an admissible polygon and since the value of all admissible polygons are equal, we conclude that any two adjacent squares carry the same number. Thus all squares carry the same number.)

9. Let K, L, M be the midpoints of sides BC, CA, AB, respectively, of triangle ABC. The points A, B, C divide the circumcircle of ABC into three arcs AB, BC, CA. Let X be the midpoint of the arc BC not containing A, let Y be the midpoint of the arc CA not containing B and let Z be the midpoint of the arc AB not containing C. Let R be the circumradius and r be the inradius of ABC. Prove that

$$r + KX + LY + MZ = 2R.$$



We present the solution by A. Robert Pargeter (England).

From the figure it is obvious that $KX = R - R\cos A$, etc. So we need to prove

$$r + 3R - R\sum \cos A = 2R, \text{ i.e., } \sum \cos A = 1 + \frac{r}{R}$$

where the summation is over the angles of triangle ABC. Now

$$\sum \cos A = \sum \frac{b^2 + c^2 - a^2}{2bc} = \sum \frac{ab^2 + ac^2 - a^3}{2abc}$$

and using $r = \Delta/s$ and $R = abc/4\Delta$, where s is the semiperimeter and Δ the area of $\triangle ABC$, we get

$$\begin{aligned} 1 + \frac{r}{R} &= 1 + \frac{4\Delta^2}{sabc} \\ &= 1 + \frac{4(s-a)(s-b)(s-c)}{abc} \\ &= 1 + \frac{(b+c-a)(c+a-b)(a+b-c)}{2abc} = \frac{X}{2abc} \end{aligned}$$

where

$$\begin{aligned} X &= 2abc + (b+c-a)(c+a-b)(a+b-c) \\ &= c^2a + c^2b - c^3 - a^3 + ab^3 + a^2b - b^3 + ca^2 + cb^2 \\ &= \sum(ab^2 + ac^2 - a^3) \end{aligned}$$

Therefore

$$\frac{X}{2abc} = 1 + \frac{r}{R} = \sum \cos A$$

as required.

49th Romania Mathematical Olympiad 1998

Selected problems from the final round

1. (7th form) Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be integers such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 + n^3 \le (2n-1)(x_1 + x_2 + \cdots + x_n) + n^2.$$

Show that

(a) $x_i \ge 0$ for $i = 1, 2, \ldots, n$.

(b) $x_1 + x_2 + \ldots + x_n + n + 1$ is not a perfect square.

We present the solution by Kiah Han Mao (Singapore). Also solved Lim Yin (Singapore).

(a) By completing squares, we have

$$\begin{aligned} & x_1^2 + \cdots + x_n^2 + n^3 \le (2n-1)(x_1 + \cdots + x_n) + n^2 \\ \Rightarrow \quad & (x_1^2 - 2x_1 n + n^2) + \cdots + (x_n^2 - 2x_n n + n^2) \\ & \quad \le (n - x_1) + \cdots + (n - x_n) \\ \Rightarrow \quad & (x_1 - n)(x_1 - n + 1) + \cdots + (x_n - n)(x_n - n + 1) \le 0. \end{aligned}$$

Since $(x_i - n)(x_i - n - 1) \ge 0$, with equality when $x_i = n, n-1$, the original inequality holds only when $x_i = n$ or $n - 1$, $i = 1, \ldots, n$.

(b) Let $S = x_1 + x_2 + \ldots + x_n + n + 1$. Then

$$(n+1)^2 > n^2 + n + 1 \ge S \ge n(n-1) + n + 1 > n^2.$$

Thus S cannot be a square.

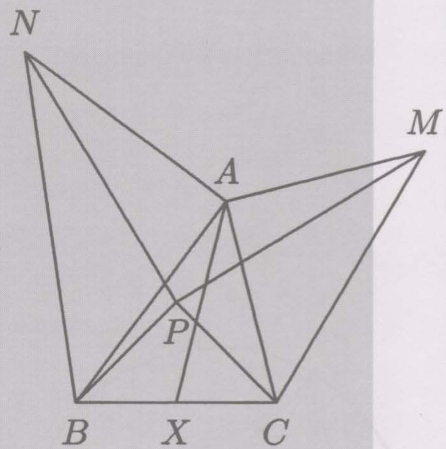
2. (7th form) Show that there is no positive integer n such that $n + k^2$ is a perfect square for at least n positive integer values of k.

We present the solution by Julius Poh Wei Quan (Raffles Junior College). Also solved by Lim Chong Jie and Lim Yin (Singapore).

Suppose $n + k^2$ is a square, then $n + k^2 \ge (k+1)^2$. Thus $k \le (n-1)/2$ and there are at most $(n-1)/2$ values of k such that $n + k^2$ is a square.

3. (7th form) In the exterior of the triangle ABC with $\angle B > 45°$, $\angle C > 45°$, one constructs the right isosceles triangles ACM and ABN such that $\angle CAM = \angle BAN = 90°$ and, in the interior of ABC, the right isosceles triangle BCP with $\angle P = 90°$. Show that MNP is a right isosceles triangle.

We present the solution by Lim Yin (Singapore). Also solved by Kiah Han Mao (Singapore).



Let X be the midpoint of BC. We have

(1) $NB = AB/\cos 45° = \sqrt{2}AB$.

(2) $BP = BX/\cos 45° = \sqrt{2}BX$.

(3) $\angle NBP = \angle NBA + \angle ABP = 45° + \angle ABP = \angle ABX$.

(4) From (1), (2) and (3), we have $\triangle NBP \sim \triangle ABX$. Similarly, $\triangle MCP \sim \triangle ACX$.

(5) Thus $\angle MPC + \angle NPB = \angle AXC + \angle AXB = 180°$. Hence $\angle MPN = 90°$.

(6) From (4), we have $NP/AX = NB/AB = \sqrt{2}$, $MP/AX = MC/AC = \sqrt{2}$. Thus $NP = MP$. This together with (5) yields that result that $\triangle MNP$ is right-angled and isosceles.

4. (9th Form) Find integers a, b, c such that the polynomial $f(x) = ax^2 + bx + c$ satisfies the equalities:

$$f(f(1)) = f(f(2)) = f(f(3)).$$

We present the solution by Julius Poh Wei Quan (Raffles Junior College). Also solved by Calvin Lin Zhiwei (Hwachong Junior College) and Lim Chong Jie (Singapore).

Since $f(x)$ is a polynomial of degree at most 2, for each k, there are at most two values of x such that $f(x) = k$. Thus we have the following cases:

(1) $f(1) = f(2)$: Then $a + b + c = 4a + 2b + c$ which implies $b = -3a$. Thus $f(x) = ax^2 - 3ax + c$ and its graph is symmetrical about the line $x = 3/2$. Now, $f(1) = f(2) = -2a + c$ and $f(3) = c$. If $c = -2a + c$, we have $a = b = 0$ and $c \in \mathbb{Z}$. If $c \neq -2a + c$, then $(-2a + c + c) = 3$ since $f(-2a + c) = f(c)$. But this is impossible as a and c are integers.

(2) $f(2) = f(3)$: This is similar to (1) and has the same solution $a = b = 0$ and $c \in \mathbb{Z}$.

(3) $f(1) = f(3)$: Then $a + b + c = 9a + 3b + c$ and $b = -4a$. So

$$f(x) = ax^2 - 4ax + c = a(x-2)^2 - 4 + c$$

and its graph is symmetrical about $x = 2$. Since $f(1) = -3a + c$ and $f(2) = -4a + c$, we have either $(-3a + c) = (-4a + c)$ or $(-3a + c) + (-4a + c) = 4$. The former gives the same solution as in (1). The latter yields $a = 2p$, $p \in \mathbb{Z}$, $b = -8p$ and $c = 2 + 7p$.

Thus the solutions are $(a, b, c) = (0, 0, p)$, or $(2p, -8p, 2 + 7p)$, $p \in \mathbb{Z}$.

5. (9th Form) Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AC-BD| \le |AB-CD|.$$

When does equality hold?

We present the solution by Kiah Han Mao (Singapore).

Let E and F be the midpoints of the diagonals AC and BD. In every quadrilateral, we have

$$AC^2+BD^2+4EF^2 = AB^2+BC^2+CD^2+DA^2.$$

(This result can be proved easily by using coordinates.) Since $ABCD$ is cyclic, by Ptolemy's Theorem, we have

$$AB\cdot CD+AD\cdot BC = AC\cdot BD.$$

Hence

$$(AC-BD)^2+4EF^2 = (AB-CD)^2+(AD-BC)^2.$$

The desired inequality would then follow if $4EF^2 \ge (AD-BC)^2$. Let M be the midpoint of AB. In ΔMEF we have $MF = AD/2$, $ME = BC/2$ and from the triangle inequality, we have $EF \ge |ME-MF|$. Hence $2EF \ge |BC-AD|$ and $4EF^2 \ge (BC-AD)^2$ as desired.

Equality holds if and only if the points M, E, F are collinear, which happens if and only if $AB||CD$, i.e., $ABCD$ is an isosceles trapezoid with $AB = CD$.

6. (10th Form) Let $n \ge 2$ be an integer and $M = \{1,2,\dots,n\}$. For every $k \in \{1,2,\dots,n-1\}$, let

$$x_k = \frac{1}{n+1}\sum_{\substack{A\subset M\\ |A|=k}} (\min A + \max A).$$

Prove that $x_1, x_2, \dots, x_{n-1}$ are integers, not all divisible by 4.

We present the solution by Kiah Han Mao (Singapore).

For each $i \in \{1,2,\dots,n\}$, there are $\binom{n-i}{k-1}$ subsets, each with k elements and contains i as the minimum element. (Note that $\binom{n}{k} = 0$ if $k > n$.) Also there are $\binom{i-1}{k-1}$ subsets, each with k elements and contains i as the maximum element. Thus

$$\begin{aligned}
x_k &= \frac{1}{n+1}\sum_{\substack{A\subset M\\ |A|=k}} (\min A + \max A)\\
&= \frac{1}{n+1}\Bigg[1\binom{n-1}{k-1}+2\binom{n-2}{k-1}+\cdots+(n+1-k)\binom{k-1}{k-1}\\
&\qquad +n\binom{n-1}{k-1}+(n-1)\binom{n-2}{k-1}+\cdots+k\binom{k-1}{k-1}\Bigg]\\
&= \binom{n}{k}
\end{aligned}$$

Thus $x_1, \ldots, x_{n-1}$ are all integers. Since $x_1 + \cdots + x_{n-1} + \binom{n}{1} + \cdots + \binom{n}{n-1} = 2^n - 2$, not all of $x_1, \ldots, x_{n-1}$ are divisible by 4.

Next is the solution by Julius Poh Wei Quan (Raffles Junior College).

For each k-element subset $A = \{a_1, \ldots, a_k\}$ with $a_i < a_j$ if $i < j$, with $a_1 = 1+p$ and $a_k = n-q$, let $B = \{b_1, \ldots, b_k\}$, where $b_i = a_i + q - p$. We have $\min A + \min B + \max A + \max B = 2n+2$. Since there are $\binom{n}{k}$ k-element subsets, we have

$$x_k = \frac{1}{n+1} \sum_{\substack{A \subset M \\ |A|=k}} (\min A + \max A) = \frac{1}{n+1}\binom{n}{k}\frac{2n+2}{2} = \binom{n}{k}.$$

41st International Mathematical Olympiad

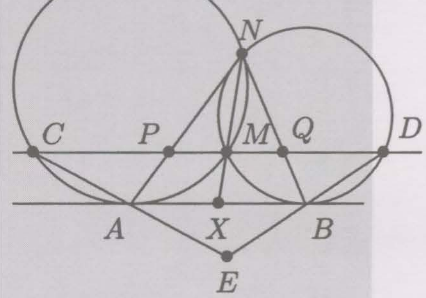
Taejon, Korea, July 2000. Day 1

1. Two circles Γ_1 and Γ_2 intersect at M and N.

Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again C and the circle Γ_2 at D.

Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q.

Show that $EP = EQ$.



Official solution: M is in fact the midpoint of PQ. To see this, extend NM meeting AB at X. Then X is the midpoint of the common tangent AB, because X being on the radical axis MN is of equal power to the two circles. As PQ is parallel to AB, M is the midpoint of PQ.

An easy diagram chasing of the angles shows that triangle EAB is congruent to triangle MAB. Hence EM is perpendicular to AB, thus perpendicular to PQ. From this it follows that $EP = EQ$.

2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

Official solution. Write $a = x/y$, $b = y/z$ and $c = z/x$ for some positive numbers x, y, z. Rewriting the inequality in terms of x, y, z we have

$$(x - y + z)(y - z + x)(z - x + y) \leq xyz.$$

Let the three factors on the left hand side be u, v, w, respectively. Since any two of u, v, w have positive sum, at most one of them is negative. If exactly one of u, v, w is negative, then the inequality holds. We are left with the case $u, v, w > 0$. By the AM-GM inequality, we have

$$\sqrt{uv} \leq \frac{u+v}{2} = x.$$

Likewise $\sqrt{vw} \leq y$, $\sqrt{wu} \leq z$. Hence $uvw \leq xyz$ as desired.

Next is the solution by Dr. Wong Yan Loi (National University of Singapore). We may assume, without loss of generality, that $a \geq 1 \geq c(>0)$. Let $d = 1/c$. Then $d \geq 1$. Substituting $b = 1/(ac)$ into the right hand side of the inequality and multiplying it out, we have

$$\begin{aligned}&(a+1/a)+(d+1/d)+(a/d+d/a)-(ad+d/(a^2)+a/(d^2))-2\\&=(a-1)(1-d)+(a-d)(d-1)/(d^2)+(d-a)(a-1)/(a^2)+1.\end{aligned}$$

We may assume $a \geq d$. Then

$$(d-a)(a-1)/(a^2) \leq 0.$$

The first two terms can be combined to get

$$\frac{(d-1)(-ad^2+d^2+a-d)}{d^2} = \frac{(d-1)^2(d-ad-a)}{d^2} \leq 0.$$

So the whole expression is ≤ 1.

Finally we have the solution by Chan Sing Chun (Singapore).

Denote the left hand side of the inequality by L. If $a - 1 + 1/b < 0$, then $a < 1$ and $b > 1$. Thus $b - 1 + 1/c$ and $c - 1 + 1/a$ are both positive, whence L is negative and the inequality holds. The same argument applies when one of the other two factors is negative. Hence forth we assume that all the three factors in L are positive. Note that $abc = 1$ implies $b(a-1+1/b) = (1/c - b+1)$, $c(b-1+1/c) = (1/a-c+1)$, $a(c-1+1/a) = (1/b-a+1)$. Thus $L = (a+1-1/b)(b+1-1/c)(c+1-1/a)$ and

$$L^2 = (a^2-(1-1/b)^2)(b^2-(1-1/c)^2)(c^2-(1-1/a)^2).$$

All these imply that

$$\begin{aligned}0 &\leq a^2-(1-1/b)^2 \leq a^2,\\0 &\leq b^2-(1-1/c)^2 \leq b^2,\\0 &\leq c^2-(1-1/a)^2 \leq c^2\end{aligned}$$

which in turn implies that

$$L^2 \leq (abc)^2 = 1 \quad \text{and} \quad L \leq 1.$$

3. Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point.

For a positive real number λ, define a *move* as follows:

> choose any two fleas, at points A and B, with A to the left of B;
>
> let the flea at A jump to the point C on the line to the right of B with $BC/AB = \lambda$.

Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to the right of M.

Official solution. We adopt the strategy to let leftmost flea jump over the rightmost flea. After k moves, let d_k denote the distance of the leftmost and the rightmost flea and δ_k denote the minimum distance between neighbouring fleas. Then $d_k \geq (n-1)\delta_k$.

After the $(k+1)$st move, there is a new distance between neighbouring fleas, namely λd_k. It can be the new minimum distance, so that $\delta_{k+1} = \lambda d_k$; and if not, then certainly $\delta_{k+1} \geq \delta_k$. In any case

$$\frac{\delta_{k+1}}{\delta_k} \geq \min\left\{1, \frac{\lambda d_k}{\delta_k}\right\} \geq \min\{1, (n-1)\lambda\}.$$

Thus if $\lambda \geq 1/(n-1)$ then $\delta_{k+1} \geq \delta_k$ for all k; the minimum distance does not decrease. So the positive of the leftmost flea keeps on shifting by steps of size not less that a positive constant, so that, eventually all the fleas will be carried as far to the right as we please.

Conversely, if $\lambda < 1/(n-1)$, we'll prove that for any initial configuration, there is a point M beyond which no flea can reach. The position of the fleas will be viewed as real numbers. Consider an arbitrary sequence of moves. Let s_k be the sum of all the numbers representing the positions of he fleas after the kth move and let w_k be the greatest of these numbers (i.e. the position of the rightmost flea). Note that $s_k \leq nw_k$. We are going to show that the sequence (w_k) is bounded.

In the $(k+1)$st move a flea from a A jumps over B, landing at C; let these points be represented by the numbers a, b, c. Then $s_{k+1} - s_k + c - a$.

By the given rules, $c - b = \lambda(b-a)$; equivalently $\lambda(c-a) = (1+\lambda)(c-b)$. Thus

$$s_{k+1} - s_k = c - a = \frac{1+\lambda}{\lambda}(c-b).$$

Suppose that $c > w_k$; the flea that has just jumped took the new rightmost position $w_{k+1} = c$. Since b was the position of some flew after the kth move, we have $b \leq w_k$ and

$$s_{k+1} - s_k = \frac{1+\lambda}{\lambda}(c-b) \geq \frac{1+\lambda}{\lambda}(w_{k+1} = w_k).$$

This estimate is valid also when $c \leq w_k$, in which case $w_{k+1} - w_k = 0$ and $s_{k+1} - s_k = c - a > 0$.

Consider the sequence of numbers

$$z_k = \frac{1+\lambda}{\lambda} w_k - s_k \quad \text{for} \quad k = 0, 1, \ldots.$$

The estimate we have just worked out shows that $z_{k+1} - z_k \leq 0$; the sequence is nonincreasing, and consequently $z_k \leq z_0$ for all k.

We have assume that $\lambda < 1/(n-1)$. Then $1 + \lambda > n\lambda$, and we can write

$$z_k = (n+\mu)w_k - s_k, \quad \text{where} \quad \mu = \frac{1+\lambda}{\lambda} - n > 0.$$

So we get the inequality $z_k = \mu w_k + (nw_k - s_k) \geq \mu w_k$. It follows that $w_k \leq z_0/\mu$ for all k. Thus the position of the rightmost flea never exceeds a constant (depending on n, λ and the initial configuration, but not on the strategy of moves). In conclusion, the values of λ, asked about, are all real numbers not less than $1/(n-1)$.

Day 2

4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card.

A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

Official solution: Suppose $1, 2 \ldots, k$, $k \geq 2$, are in box 1, and $k+1$ in box 2 and m is the smallest number in box 3. Then $m-1$ is either in box 1 or 2. But it can't be in box 1 for $(m) + (k) = (m-1) + (k+1)$, but it can't be in box 2 either as $(m) + (1) = (m-1) + 2$. Thus we conclude that 1 and 2 are in different boxes. So we assume that 1 is in box 1, and $2, \ldots, k$, $k \geq 2$ are in box 2, $k+1$ not in box 2 and m is the smallest number in box 3. If

$m > k+1$, then $k+1$ is in box 1. Also $m-1$ is not in box 1 as $(m-1)+(2) = (m)+(1)$. Thus $m-1$ is in box 2. This is not possible as $(m)+(k) = (m-1)+(k+1)$. Thus $m = k+1$. If $k = 2$, we have 1, 2, 3 in different boxes. Since a in box 1, $a+1$ in box 2, $a+2$ box 3 imply that $a+3$ is in box 1. We have box i contains all the numbers congruent to $i \pmod 3$. This distribution clearly works since $a \equiv i, b \equiv j \pmod 3$ imply $a+b \equiv k \pmod 3$ where $k \not\equiv i, j \pmod 3$.

Now suppose that $k \geq 3$. We conclude that $k+2$ can't be included in any box. Thus $k = 99$. We see that this distribution also works.

Hence there are altogether 12 ways.

Second solution: Consider 1, 2 and 3. If they are in different boxes, then 4 must be in the same box as 1, 5 in the same box as 2 and so on. This leads to the solution where all numbers congruent to each other mod 3 are in the same box.

Suppose 1 and 2 are in box 1 and 3 in box 2. Then 4 must be in box 1 or 2. In general, if $k(\geq 4)$ is in either box 1 or 2, then $k+1$ also must be in box 1 or 2. Thus box 3 is empty which is impossible.

Similarly, it is impossible for 1 and 3 to be in box 1 and 2 in box 2.

Thus we are left with the case where 1 is in box 1 and 2 and 3 in box 2. Suppose box 2 contains $2, \ldots k$, where $k \geq 3$, but does not contain $k+1$ and m is the smallest number in box 3. Then $m > k$. If $m > k+1$, then $k+1$ must be box 1 and we see that no box can contain $m-1$. Thus $m = k+1$. If $k < 99$, we see that no box can contain $k+2$. Thus $k = 99$. It is easy to see that this distribution works. Thus there altogther 12 ways.

Third solution which is official:

We show that the answer is 12. Let the colour of the number i be the colour of the box which contains it. In the sequel, all numbers considered are assumed to be integers between 1 and 100.

Case 1. There is an i such that i, $i+1$, $i+2$ have three different colours, say **rwb**. Then, since $i+(i+3) = (i+1)+(i+2)$, the colour of $i+3$ can be neither **w**(the colour of $i+1$) nor **b**(the colour of $i+2$). It follows that $i+3$ is **r**. Using the same argument, we see that the next numbers are also **rwb**. In fact the argument works backwards as well: the previous three numbers are also **rwb**. Thus we have 1, 2 and 3 in different boxes and two numbers are in the same box if there are congruent mod 3. Such an arrangement is good as $1+2$, $2+3$ and $1+3$ are all different mod 3. There are 6 such arrangements.

Case 2. There are no three neighbouring numbers of different colours. Let 1 be red. Let i be the smallest non-red number, say white. Let the smallest blue number be k. Since there is no **rwb**, we have $i+1<k$.

Suppose that $k<100$. Since $i+k=(i-1)+(k+1)$, $k+1$ should be red. However, in view if $i+(k+1)=(i+1)+k$, $i+1$ has to be blue, which draws a contradiction to the fact that the smallest blue is k. This implies that k can only be 100.

Since $(i-1)+100=i+99$, we see that 99 is white. We now show that 1 is red, 100 is blue, all the others are white. If $t>1$ were red, then in view of $t+99=(t-1)+100$, $t-1$ should be blue, but the smallest blue is 100.

So the colouring is **rww ... wwb**, and this is indeed good. If the sum is at most 100, then the missing box is blue; if the sum is 101, then it is white and if the sum is greater than 101, then it is red. The number of such arrangements is 6.

5. Determine whether or not there exists n such that

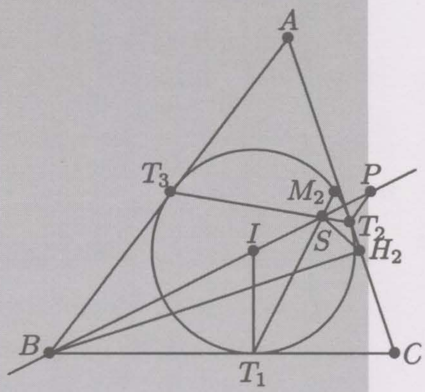
> n is divisible by exactly 200 different prime numbers and 2^n+1 is divisible by n.

Official solution: The answer is yes and we shall prove it proving a more general statement: *For each $k\in\mathbb{N}$, there exists $n=n(k)\in\mathbb{N}$ such that $n\mid 2^n+1$, $3\mid n$ and n has exactly k prime factors.* We shall prove it by induction on k.

We have $n(1)=3$. We then assume for some $k\geq 1$, there exists $n=n(k)$ with the desired properties. Then n is odd. Since $2^{3n}+1=(2^n+1)(2^{2n}-2^n+1)$ and 3 divides the second factor, we have $3n\mid 2^{3n}+1$. For any positive odd integer m, we have $2^{3n}+1\mid 2^{3nm}+1$. Thus if p is prime number such that $p\nmid n$ and $p\mid 2^{3n}+1$, then $3np|2^{3np}+1$ and $n(k+1)=3pn$ has the desired properties. Thus the proof would be complete if we can find such a p. This is achieved by the following lemma:

> **Lemma.** For any integer $a>2$ such that $3\mid a+1$, there exists a prime number p such that $p\mid a^3+1$ but $p\nmid a+1$.
>
> *Proof.* Assume that this is false for a certain integer $a>2$. Since $a^3+1=(a+1)(a^2-a+1)$, each prime divisor of a^2-a+1 divides $a+1$. Since $a^2-a+1=(a+1)(a-2)+3$, we conclude that a^2-a+1 is a power of 3. Since $a+1$ and $a-2$ are both multiples of 3, we conclude that $9\nmid a^2-a+1$. This gives a contradiction as $a^2-a+1>3$ for $a>2$.

6. Let AH_1, BH_2, CH_3 be the altitudes of an acute-angled triangle ABC. The incircle of the triangle ABC touches the sides BC, CA, AB at T_1, T_2, T_3, respectively. Let the lines ℓ_1, ℓ_2, ℓ_3 be the reflections of the lines H_2H_3, H_3H_1, H_1H_2 in the lines T_2T_3, T_3T_1, T_1T_2, respectively.



Prove that ℓ_1, ℓ_2, ℓ_3 determine a triangle whose vertices lie on the incircle of the triangle ABC.

Official solution: Let M_1, M_2, M_3 be the reflections of T_1, T_2, T_3 across the bisectors of $\angle A, \angle B, \angle C$, respectively. The points M_1, M_2, M_3 obviously lie of the incircle of $\triangle ABC$. We prove that they are the vertices of the triangle formed by the images in question, which settle the claim.

By symmetry, it suffices to show that the reflection l_1 of H_1H_2 in T_2T_3 passes through M_2. Let I be the incentre of $\triangle ABC$. Note that T_2 and H_2 are always on the same side of BI, with T_2 closer to BI than H_2. We consider only the case when C is on this same side of BI, as in the figure (minor modifications are needed if C is on the other side).

Let $\angle A = 2\alpha, \angle B = 2\beta, \angle C = 2\gamma$.

Claim 1: the mirror image of H_2 with respect to T_2T_3 lies on the line BI.

Proof of claim 1: Let $\ell \perp T_2T_3$, $H_2 \in \ell$. Denote by P and S the points of intersection of BI with ℓ and BI with T_2T_3. Note that S lies on both line segments T_2T_3 and BP. It is sufficient to prove that $\angle PSH_2 = 2\angle PST_2$. We have $\angle PST_2 = \angle BST_3$ and by the external angle theorem,

$$\angle BST_3 = \angle AT_3S - \angle T_3BS = (90^\circ - \alpha) - \beta = \gamma.$$

Next $\angle BST_1 = \angle BST_3 = \gamma$ by symmetry across BI. Note that C and S are on the same side of IT_1, since $\angle BT_1S = 90^\circ + \alpha > 90^\circ$. Then, in view of the equalities $\angle IST_1 = \angle ICT_1 = \gamma$, the quadrilateral SIT_1C is cyclic, so $\angle ISC = \angle IT_1C = 90^\circ$. Hence $\angle BSC = \angle BH_2C$, and hence the quadrilateral BCH_2S is cyclic. It shows that $\angle PSH_2 = \angle C = 2\gamma = 2\angle PST_2$, as needed. This completes of the proof of the claim.

Note that the proof of the claim also gives

$$\angle BPT_2 = \angle SH_2T_2 = \beta,$$

by symmetry across T_2T_3 and because the quadrilateral BCH_2S is cyclic. Then, since M_2 is the reflection of T_2 across BI, we obtain $\angle BPM_2 = \angle BPT_2 = \beta = \angle CBP$, and so PM_2 is parallel to BC. To prove that M_2 lies on ℓ_1, it now suffices to show that ℓ_1 is also parallel to BC.

Suppose $\beta \neq \gamma$; let the line CB meet H_2H_3 and T_2T_3 at D and E, respectively. (Note that D and E lie on the line BC on the same side of the segment BC.) An easy angle computation gives $\angle BDH_3 = 2|\beta - \gamma|$, $\angle BET_3 = |\beta - \gamma|$, and so the line ℓ_1 is indeed parallel to BC. The proof is now complete.