Problems Corner

Contest

1. Prizes in the form of book vouchers will be awarded to the first received best solution(s) submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

3. Solutions should be typed and sent to:
The Editor,
Mathematics Medley,
c/o Department of Mathematics,
National University of Singapore,
2 Science Drive 2,
Singapore 117543;
and should arrive before
28 February 2001.

4. The Editor's decision will be final and no correspondence will be entertained.

Problem 1

Find all positive integers n such that $n + s(n) = 2001$, where $s(n)$ is the sum of all the digits of n.

PRIZE
one $100 book voucher

Problem 2

Prove that for any positive real numbers a, b, c,

$$\frac{a}{10b+11c} + \frac{b}{10c+11a} + \frac{c}{10a+11b} \geq \frac{1}{7}.$$

PRIZE
one $100 book voucher

Problem 1.

Find all pairs of positive integers (m, n) such that $m^2 + 2000 = 6^n$.

(*Prize* One \$100 book Voucher)

SOLUTIONS

Solutions to the problems in volume 27 No. 1 August 2000

SOLUTIONS to Problem 1

by **Chairmaine Sin Jin Min**
Raffles Girls' School Secondary School
Class 1/13

$m^2 + 2000 = 6^n = 2^n 3^n$.
$\therefore n \geq 5$ as $m^2 > 0$.

$2^4 | (m^2 + 2000), 2^4 | 2000 \implies 2^4 | m^2 \implies 2^2 | m$.
$\therefore$ we write m as $4x$.

$16x^2 + 2000 = 2^n 3^n$.
$x^2 + 125 = 2^{n-4} 3^n$.
$2^{n-4} 3^n = 3^4 6^{n-4} \equiv 6 \pmod{10}$.
$\therefore x^2 \equiv 1 \pmod{10}$.
$x \equiv 1, 9 \pmod{10}$.

If $x \equiv 1 \pmod{10}$, then by letting $x = 10y + 1$, we have $50y^2 + 10y + 63 = 2^{n-5} 3^n$. But n cannot be more than 5 as $50y^2 + 10y + 63$ is odd, whereas $2^{n-5} 3^n$ is even for $n > 5$. Therefore $n = 5$.

If $x \equiv 9 \pmod{10}$, then by letting $x = 10y + 9$, we have $50y^2 + 90y + 103 = 2^{n-5} 3^n$. Similarly, n cannot be more than 5 as $50y^2 + 90y + 103$ is odd, whereas $2^{n-5} 3^n$ is even for $n > 5$. Therefore $n = 5$.

When $n = 5$, we have $m = 76$.

Editor's note: A slightly different argument is as follow. Suppose $n \geq 6$. Then $m^2 = 6^n - 2000 = 2^n 3^n - 2^4 5^3 = 2^4(2^{n-4} 3^n - 5^3)$. Hence, $2^{n-4} 3^n - 5^3$ is also a perfect square. However, $2^{n-4} 3^n - 5^3 \equiv 3$ modulo 4. This contradicts the fact that the square of a number is either congruent to 0 or 1 modulo 4. Therefore, $n \leq 5$. Now a direct calculation shows that $(m, n) = (76, 5)$ is the only solution.

Solved also by Colin Tan Weiyu, Raffles Institution, Class 3M, and Gideon Tan Guangyuan, Raffles Institution, Class 3L. The prize went to Charmaine Sia Jia Min.

Problem 2.

ABCD is a quadrilateral inscribed in a circle with centre at O, and P is the intersection of AC and BD. Let O_1, O_2, O_3 and O_4 be the circumcentres of the triangles PAB, PBC, PCD and PDA respectively. Prove that the lines OP, O_1O_3 and O_2O_4 are concurrent.

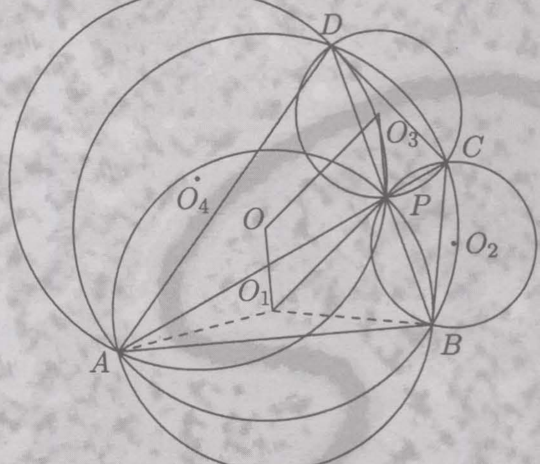
(*Prize* One $100 book Voucher)

SOLUTIONS

Solutions to the problems in volume 27 No. 1 August 2000

SOLUTIONS to Problem 2

by **editor**



First, let's prove that PO_1 is parallel to OO_3. To do so, let's prove that O_1P is perpendicular to CD. This can be seen as follow. Join O_1A and O_1B. Extend O_1P meeting CD at X. For simplicity, X is not shown in the diagram. We need to show that $\angle CPX + \angle PCD = 90°$.

Note that $\angle CPX = \angle O_1PA = \angle O_1AP$ and $\angle AO_1P = 2\angle PBA = 2\angle PCD$. Hence, we deduce from the angle sum of the triangle AO_1P that $\angle CPX + \angle PCD = \frac{1}{2}(\angle AO_1P + \angle O_1PA + \angle O_1AP) = 90°$. As OO_3 is also perpendicular to to CD, we have PO_1 is parallel to OO_3.

Similarly, PO_3 is also parallel to OO_1. This means that PO_1OO_3 is a parallelogram so that PO and O_1O_3 bisect each other. Similarly, PO_2OO_4 is a parallelogram so that PO and O_2O_4 bisect each other. That means the three lines PO, O_1O_3, O_2O_4 concur at their common midpoint.

Editor's note: Solved also by Colin Tan Weiyu, Raffles Institution, Class 3M, and Gideon Tan Guangyuan, Raffles Institution, Class 3L. The prize was shared equally between by Colin Tan Weiyu and Gideon Tan Guangyuan.