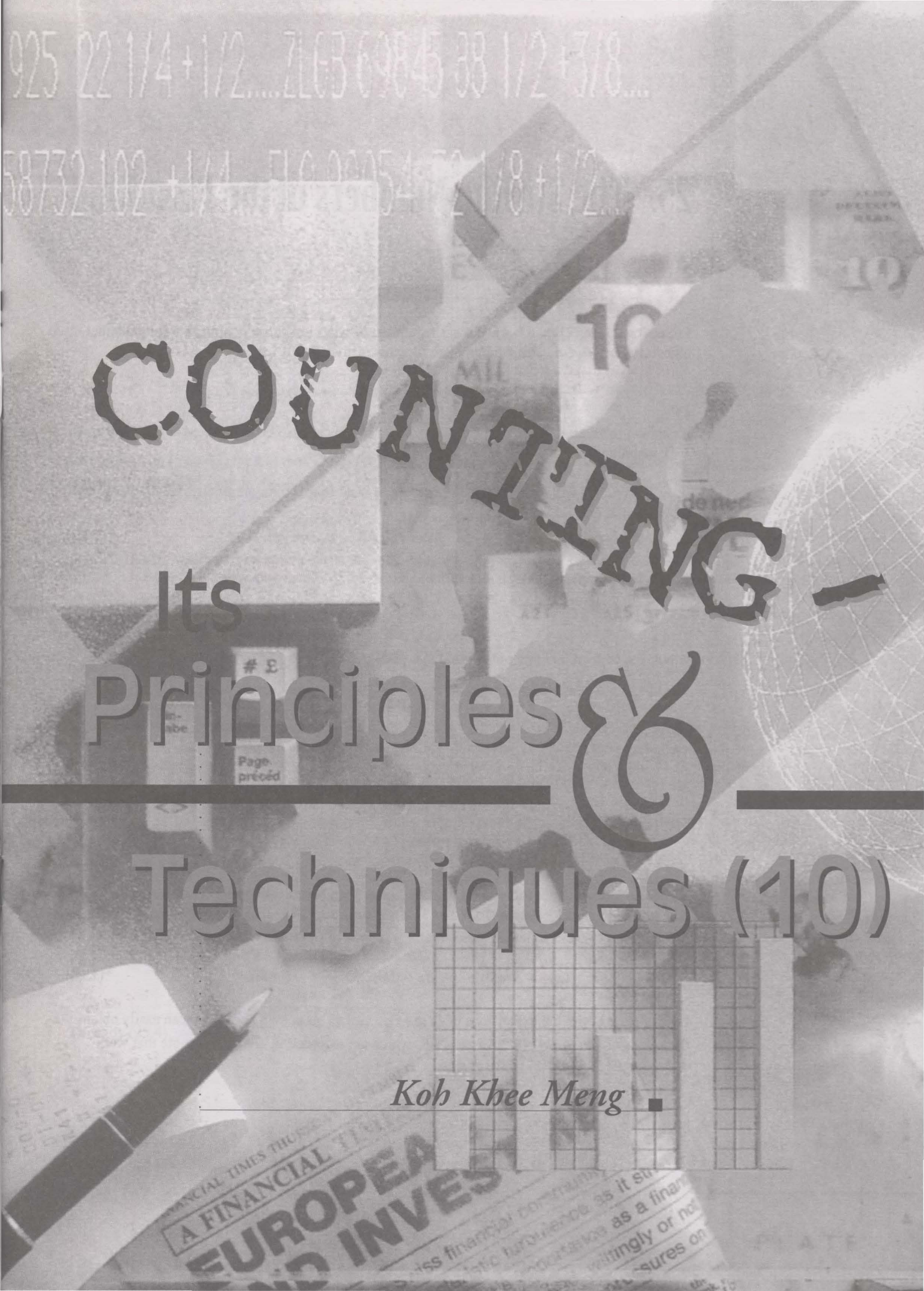


COUNTING – Its Principles & Techniques (10)

Koh Khee Meng

25. The Stirling Numbers of the First Kind

In Section 4 (see [2]), we learned that the number of ways to choose m objects from n distinct objects, where $m \le n$, and arrange them in a row is given by

$$n(n-1)(n-2)\ldots(n-(m-1)). \tag{25.1}$$

As shown in Section 19 (see [7]), the expression (25.1) can also be interpreted as the number of $1-1$ mappings from the set $\{1,2,\ldots,m\}$ to the set $\{1,2,\ldots,n\}$.

As (25.1) will be mentioned very often in what follows, for simplicity, we may denote it by $[n]_m$; that is,

$$[n]_m = n(n-1)(n-2)\ldots(n-(m-1)). \tag{25.2}$$

Let us replace 'n' by a real variable 'x' in (25.2). Then we have

$$[x]_m = x(x-1)(x-2)\ldots(x-(m-1)), \tag{25.3}$$

which can be regarded as a polynomial in x of degree m. For instance,

$$\begin{aligned}
&[x]_1 = x,\\
&[x]_2 = x(x-1) = -x + x^2,\\
&[x]_3 = x(x-1)(x-2) = 2x - 3x^2 + x^3,\\
&[x]_4 = x(x-1)(x-2)(x-3) = -6x + 11x^2 - 6x^3 + x^4,\\
&[x]_5 = x(x-1)(x-2)(x-3)(x-4) = 24x - 50x^2 + 35x^3 - 10x^4 + x^5,\\
&\text{etc.}
\end{aligned}$$

Just like what we did above, we shall express the polynomial $[x]_m$ in increasing order of powers of x. The following question arises naturally: what can be said about the coefficient of x^k in the expansion of $[x]_m$, where $0 \le k \le m$? It is clear from (25.3) that this coefficient depends on both 'm' and 'k'; and so let us, at this moment, denote it by $s(m,k)$. Thus, we have:

$$[x]_m = s(m,0) + s(m,1)x + s(m,2)x^2 + \cdots + s(m,m)x^m. \tag{25.4}$$

By comparing with the expansions of $[x]_1$, $[x]_2$, ..., $[x]_5$ as shown above, we can easily obtain the values of $s(m,k)$, where $0 \le k \le m \le 5$, which are recorded in Table 25.1 (note that we define $s(0,0)$ to be 1).

m \ k	0	1	2	3	4	5
0	1					
1	0	1				
2	0	−1	1			
3	0	2	-3	1		
4	0	-6	11	-6	1	
5	0	24	-50	35	-10	1

Table 25.1. The values of $s(m,k)$, $0 \le k \le m \le 5$

If follows from (25.3) and (25.4) that $s(m,0) = 0$ and $s(m,m) = 1$ for all $m \ge 1$. Also, the sequence of numbers:

$$s(m,1),\ s(m,2),\ \ldots,\ s(m,m)$$

alternate in sign with $s(m,1)$ positive when and only when m is odd.

Problem 25.1. Find the values of $s(6,k)$, where $1 \le k \le 6$.

Problem 25.2. Show that for $m \ge 2$,

(i) $s(m,1) = (-1)^{m+1}(m-1)!$;

(ii) $s(m,m-1) = -\binom{m}{2}$.

How are we going to evaluate $s(m,k)$? We see from above (including Problem 25.2) that when $k = 0$, 1 or $k = m-1$, m, the values of $s(m,k)$ can be computed by simple formulas. For general 'k', there is a 'recursive' way to evaluate $s(m,k)$ that we shall now present.

By comparing $[x]_m$ and $[x]_{m-1}$ by (25.3), we have

$$[x]_m = \underbrace{x(x-1)\ldots(x-(m-2))}(x-(m-1))$$

$$= [x]_{m-1}\ (x-(m-1)).$$

Thus, by (25.4),

$$
\begin{aligned}
& s(m,0)+s(m,1)x+s(m,2)x^2+\cdots+s(m,m)x^m \\
&=[x]_m \\
&=(x-(m-1))\,[x]_{m-1} \\
&=(x-(m-1))\,(s(m-1,0)+s(m-1,1)x+s(m-1,2)x^2+\cdots+s(m-1,m-1)x^{m-1}) \\
&=-(m-1)\,s(m-1,0)+(s(m-1,0)-(m-1)\,s(m-1,1))x \\
&\quad +(s(m-1,1)-(m-1)\,s(m-1,2))x^2+\cdots+s(m-1,m-1)x^m .
\end{aligned}
$$

Hence, by equating the coefficients of x^k on the two sides of the above equality, we have

$$
\begin{aligned}
s(m,0) &= \qquad\qquad -(m-1)\,s(m-1,0), \\
s(m,1) &= s(m-1,0)-(m-1)\,s(m-1,1), \\
s(m,2) &= s(m-1,1)-(m-1)\,s(m-1,2), \\
&\ \ \vdots
\end{aligned}
$$

and, in general,

$s(m,k)=s(m-1,k-1)-(m-1)\,s(m-1,k)$ with the conditions that $s(r,0)=0$ for $r \geq 1$ and $s(r,r)=1$ for all $r \geq 0$. (25.5)

As was pointed out before, the value of $s(m,k)$ depends on two parameters : m and k. In (25.5), we observe that the value of $s(m,k)$ is expressed in terms of the values of $s(m-1,k-1)$ and $s(m-1,k)$, where the values of the parameters do not exceed those in $s(m,k)$. Thus, we can evaluate $s(m,k)$ if we know the values of $s(p,q)$ where $p \leq m$ and $q \leq k$. For instance, when $(m,k)=(6,3)$, by (25.5),

$$s(6,3)=s(5,2)-5\,s(5,3).$$

Checking from Table 25.1, we have $s(5,2)=-50$ and $s(5,3)=35$. Thus

$$s(6,3)=-50-5(35)=-225.$$

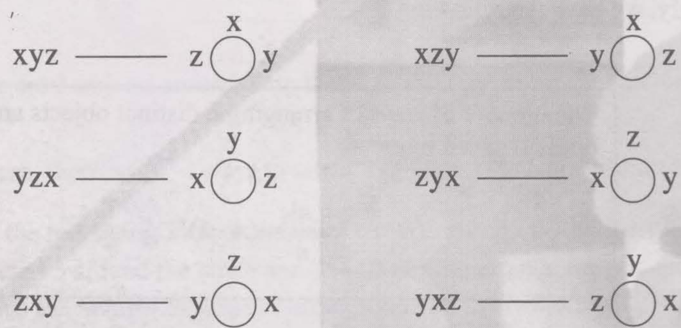
In Sections 23 and 24 (see [9]), we introduced the notion of 'recurrence relations', such as $a_n=a_{n-1}+a_{n-2}$ and $a_n=2a_{n-1}+1$, where the value of a_n is expressed in terms of the values of a_r's where $r<n$. The relation (25.5) is also regarded as a recurrence relation, but is more complicated as it involves two parameters.

The numbers $s(m,k)$ are called the *Stirling numbers of the first kind* in honour of the Scottish mathematician James Stirling (1692 – 1770). Inspired by the theory on plane curves due to Isaac Newton, Stirling worked on its extensions and published in 1730 his most influential work *Methodus Differentialis*, where the numbers $s(m,k)$ were introduced and the famous approximation formula $n! \approx \sqrt{2\pi n}\left(\dfrac{n}{e}\right)^n$ was found.

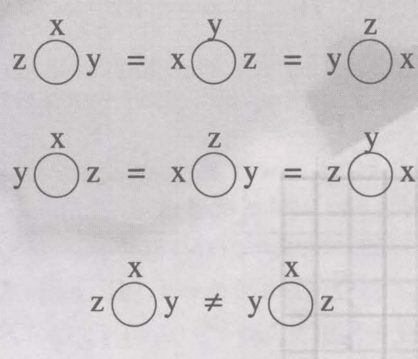
26. Combinatorial Interpretation

The Stirling number $s(m,k)$ was defined as the coefficient of x^k in the expansion of $[x]_m$, which is purely algebraic in nature. Does it have any combinatorial interpretation? The answer is yes, and we shall now present one.

Before we can really appreciate the interpretation, let us first introduce the notion of 'circular permutation'. In Section 4 (see [2]), we learned that the number of ways to arrange n *distinct* objects in a *row* is given by $n!$. If we wish to arrange them around a *circle*, in how many ways can this be done? Take, for example, $n = 3$ and suppose that X, Y and Z are the three distinct objects to be arranged. The 3! (=6) ways of row arrangement are shown on the left of Figure 26.1 while their corresponding circular arrangements are shown on the right.



mean by saying that two arrangements are regarded as the same. If we look at the 6 circular arrangements of Figure 26.1, we may notice that among the first three (resp., the last three), any one is a rotation of every other (i.e., the relative positions of X, Y and Z are the same under rotation). Let us agree and say that these arrangements are the same and that there are no others. Thus



It follows that there are only '2' different ways of arranging 3 distinct objects around a circle.

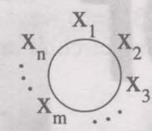
Suppose now there are n distinct objects $X_1, X_2, \ldots, X_n$ to be arranged. Then there are $n!$ ways to arrange them in a row. From what we discussed above, we see that if we start with a row arrangement, say,

$$X_1 X_2 \ldots X_n,$$

then we can generate the following n row arrangements (each of them obtained form the preceding one by putting the first object at the end):

$$\begin{cases} X_1 X_2 \ldots X_{n-1} X_n, \\ X_2 X_3 \ldots X_n X_1, \\ X_3 X_4 \ldots X_1 X_2, \\ \quad\vdots \\ X_n X_1 \ldots X_{n-2} X_{n-1}, \end{cases}$$

and all these correspond to one and only one circular arrangement:

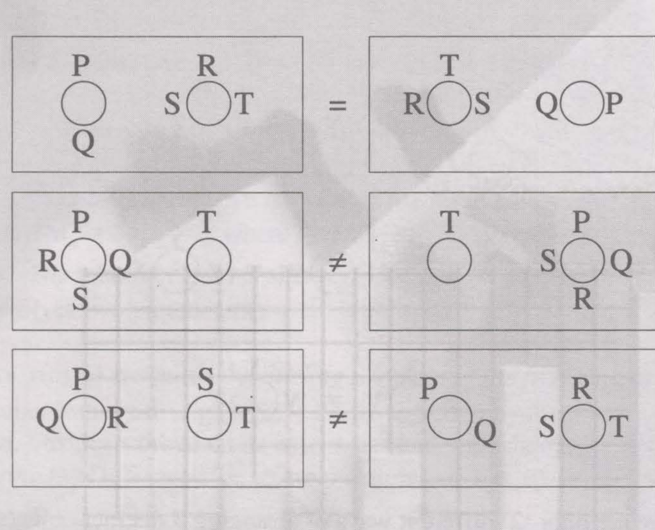


Accordingly, we have the following:

> The number of ways of arranging n distinct objects around a circle is given by
>
> $$\frac{n!}{n} = (n-1)!. \qquad (26.1)$$

Let us proceed further to study a variation of circular arrangements. Suppose now there are, say, 5 *distinct* objects to be arranged around, say, 2 *identical* circles with at least one object at each circle (see Problem 24.3 [9]). In how many ways can this be done? Again, before we move on, let us agree on what we mean that two ways of such arrangements are the same.

We define, for instance, that



while

With this clarification, we are now ready to solve the problem. There are two ways to split 5 distinct objects into 2 nonempty groups; namely,

(i) $4+1$, (ii) $3+2$.

Case (i). There are 4 objects around a circle and 1 object around another circle.

In this case, these are $\binom{5}{4}$ ways to select 4 objects from 5 distinct ones and put them around one circle (and, of course, the remaining one is on the other circle). By (26.1), there are $(4-1)!$ ways to arrange the selected '4' around the circle. Thus the number of ways of arrangement is, by (MP),

$$\binom{5}{4}(4-1)!=5\cdot 3!=30.$$

Case (ii). There are 3 objects around a circle and 2 objects around another circle.

In this case, there are $\binom{5}{3}$ ways to select 3 objects from 5 and put them around a circle (and, of course, the remaining 2 are at the other circle). By (26.1), there are $(3-1)!$ way to arrange the selected 3 around the circle and $(2-1)!$ ways to arrange the remaining 2 around the other circle. Thus the number of ways of arrangement is, by (MP),

$$\binom{5}{3}(3-1)!\,(2-1)!=20.$$

Finally, by (AP), the required number of ways of arrangement is given by $30 + 20 = 50$.

We thus conclude that the number of ways of arranging 5 distinct objects around 2 identical circles with at least one object at each circle is '50'.

Note that '50' is related to a Stirling number of the first kind. Indeed, $s(5,2)=-50$ (m = 5 corresponds to 5 objects and k = 2 corresponds to 2 circles).

Problem 26.1. Show that the number of ways of arranging 6 distinct objects around 3 identical circles with at least one object at each circle is given by 225.

For convenience, let us denote by $s^*(m,k)$ with $k \le m$ the number of ways of arranging *m distinct* objects around *k identical* circles with at least one object at each circle. Thus, as shown above, $s^*(5,2)=50=|s(5,2)|$; and also by comparing the answers of Problems 25.1 and 26.1, we have $s^*(6,3)=225=|s(6,32)|$, where $|x|$ denotes the absolute value of the real number x. We define $s^*(0,0)=1$. Clearly, $s^*(m,0)=0$ and $s^*(m,1)=(m-1)!$ by (26.1). Our aim is to show that, indeed, $s^*(m,k)=|s(m,k)|$.

The result (25.5) provides us with a recurrence relation for the numbers $s(m,k)$. In what follows, we shall establish a corresponding recurrence relation for $s^*(m,k)$.

> For $m,k \in \mathbf{N}$ with $k \le m$,
> $$s^*(m,k) = s^*(m-1,k-1) + (m-1)s^*(m-1,k). \qquad (26.2)$$

Let us give a combinatorial argument to see why (26.2) holds. The number $s^*(m,k)$ counts the number of ways of arranging m distinct objects, say $X_1, X_2, \ldots, X_m$ around k identical circles with at least one object at each circle. Let us fix one of the m objects, say X_m. Clearly, in any such arrangement, either (i) X_m is the 'only' object at a circle or (ii) X_m is mixed with others at a circle. We now count $s^*(m,k)$ by splitting our consideration into the above 2 cases.

Case (i). X_m is the only object at a circle.

In this case, the remaining objects $X_1, X_2, \ldots, X_{m-1}$ are arranged around $k-1$ identical circles with at least one object at each circle. By definition, there are $s^*(m-1,k-1)$ ways.

Case (ii). X_m is mixed with others at a circle.

In this case, we can accomplish the task by first arranging the objects $X_1, X_2, \ldots, X_{m-1}$ around k circles with at least one object at each circle, and then place X_m in one of the circles. By definition, there are $s^*(m-1,k)$ ways to perform the first step. How many ways are there for the 2nd step? After arranging $m-1$ objects around the circles, X_m can be placed at any of the $m-1$ spaces (say, the immediate left of an object), and so there are m – 1 ways to do so. Thus, by (MP), there are $(m-1)s^*(m-1,k)$ ways in this case. Finally, by (AP), we arrive at the result (26.2).

With the help of (25.5) and (26.2), we shall now see why $s^*(m,k) = |s(m,k)|$ for all $m,k \in \mathbf{N}$ with $k \le m$. Note that for all $m \ge 1$, $s^*(m,m) = 1 = s(m,m)$ and $s^*(m,1) = (m-1)! = |s(m,1)|$. Also, as $s(m-1, k-1)$ and $s(m-1, k)$ are *different in sign*, we have

$$|s(m-1,k-1) - (m-1)s(m-1,k)| = |s(m-1,k-1)| + (m-1)|s(m-1,k)|. \qquad (26.3)$$

Consider $(m,k) = (3,2)$. Observe that

$$\begin{aligned}
|s(3,2)| &= |s(2,1) - 2s(2,2)| && \text{(by (25.5))}\\
&= |s(2,1)| + 2|s(2,2)| && \text{(by (26.3))}\\
&= s^*(2,1) + 2s^*(2,2) && \text{(see above)}\\
&= s^*(3,2). && \text{(by (26.2))}
\end{aligned}$$

When $(m,k) = (4,2)$, we have

$$\begin{aligned} |s(4,2)| &= |s(3,1) - 3s(3,2)| && \text{(by (25.5))} \\ &= |s(3,1)| + 3|s(3,2)| && \text{(by (26.3))} \\ &= s^*(3,1) + 3s^*(3,2) && \text{(see above)} \\ &= s^*(4,2). && \text{(by (26.2))} \end{aligned}$$

When $(m,k) = (4,3)$, we have

$$\begin{aligned} |s(4,3)| &= |s(3,2) - 3s(3,3)| && \text{(by (25.5))} \\ &= |s(3,2)| + 3|s(3,3)| && \text{(by (26.3))} \\ &= s^*(3,2) + 3s^*(3,3) && \text{(see above)} \\ &= s^*(4,3). && \text{(by (26.2))} \end{aligned}$$

If we proceed in this manner by following the ordering, say, (m,k) = (3,2), (4,2), (4,3), (5,2), (5,3), (5,4), (6,2), (6,3), (6,4), (6,5), (7,2), (7,3), …, we shall always find that

$$|s(m,k)| = s^*(m,k). \qquad (26.3)$$

Let us explain why (26.3) holds for all (m,k) where $1 \le k \le m$ with the help of Figure 26.2. We have already verified that (26.3) holds when $k = 1$ and $k = m$. This is indicated at the entries (m,k) enclosed by rectangles in Figure 26.2. In the above process, the key tools we employ are the recurrence relations (25.5) and (26.2) (and, of course, (26.3) also). We then start with (m,k) = (3,2). Using the verified results for (2,1) and (2,2), and applying (25.5) and (26.2), we show that (26.3) holds for (3,2). This fact is indicated by the two arrows pointing to entry (3,2) from entries (2,1) and (2,2) in Figure 26.2. We then proceed to (m,k) = (4,2). Using the verified results for (3,1) and (3,2), and applying (25.5) and (26.2), we show that (26.3) holds for (4,2). Again, this is indicated by the arrows pointing to (4,2) in the figure. Thus, following the ordering of (m,k) as fixed above and the arrows pointing to the corresponding entries in Figure 26.2, we see that the result (26.3) is indeed valid for each (m,k) with $1 \le k \le m$.

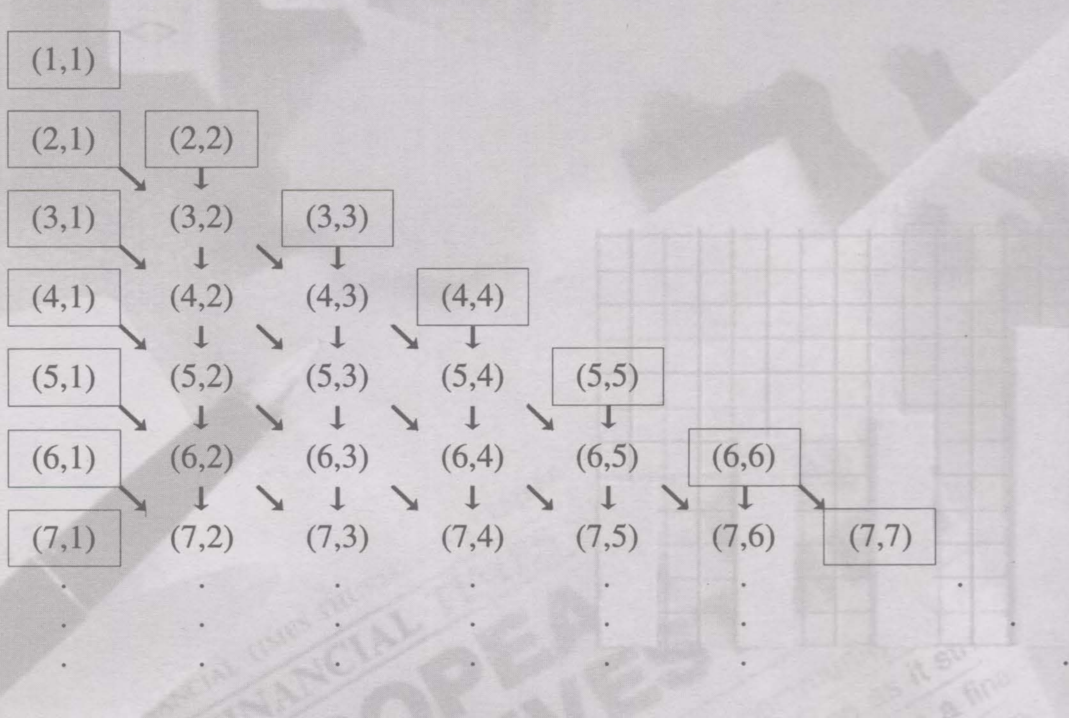


Figure 26.2

References.

[1] K. M. Koh and B. P. Tan, *Counting-Its Principles and Techniques (1)*, Mathematical Medley Vol. 22 March (1995) 8 – 13.

[2] K. M. Koh and B. P. Tan, *Counting-Its Principles and Techniques (2)*, Mathematical Medley Vol. 22 September (1995) 47 – 51.

[3] K. M. Koh and B. P. Tan, *Counting-Its Principles and Techniques (3)*, Mathematical Medley Vol. 23 March (1996) 9 – 14.

[4] K. M. Koh and B. P. Tan, *Counting-Its Principles and Techniques (4)*, Mathematical Medley Vol. 23 September (1996) 44 – 50.

[5] K. M. Koh and B. P. Tan, *Counting-Its Principles and Techniques (5)*, Mathematical Medley Vol. 24 March (1997) 8 – 12.

[6] K. M. Koh and B. P. Tan, *Counting-Its Principles and Techniques (6)*, Mathematical Medley Vol. 24 September (1997) 46 – 48.

[7] K. M. Koh, *Counting-Its Principles and Techniques (7)*, Mathematical Medley Vol. 25 March (1999) 63 - 74.

[8] K. M. Koh, *Counting-Its Principles and Techniques (8)*, Mathematical Medley Vol. 26 July (1999) 4 - 8.

[9] K. M. Koh, *Counting-Its Principles and Techniques (9)*, Mathematical Medley Vol. 26 Dec (1999) 80 – 90.