COMPETITION CORNER

In this issue we publish the problems of the Austria-Polish Mathematics Competition 1998, the South African Mathematical Olympiad 1999, the 49th Romanian National Mathematical Olympiad 1998, as well as the 41st International Mathematical Olympiad which was held in July 2000 at Taejon, Korea. Please send your solutions of these Olympiads to the address given above. All correct solutions will be acknowledged. We also present solutions of the 12th Nordic Mathematical Contest 1998, the 1st Japan Mathematical Olympiad 1991, the Georgian Mathematical Olympiad 1997 and the 40th International Mathematical Olympiad 1999.

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• Problems •

South African Mathematical Olympiad, 1999

Third round

1. How many non-congruent triangles with integer sides and perimeter 1999 can be constructed?

2. A, B, C and D are points on a given straight line, in that order. Construct a square $PQRS$, with all of P, Q, R and S on the same side of AD, such that A, B, C and D lie on PQ, SR, QR and PS produced, respectively.

3. The bisectors of angle BAD in the parallelogram $ABCD$ intersects the lines BC and CD at the point K and L, respectively. Prove that the centre of the circle passing through the points C, K and L lies on the circle passing through the points B, C and D.

4. The sequence $L_1, L_2, L_3, \ldots$ is defined by

$$L_1 = 1,\ L_2 = 3,\ L_n = L_{n-1} + L_{n-2} \text{ for } n > 2,$$

so the six terms are $1, 3, 4, 7, 11, 18$. Prove that $L_p - 1$ is divisible by p if p is prime.

5. Let S be the set of all rational numbers whose denominators are powers of 3. Let a, b and c be given non-zero real numbers. Determine all real-valued functions f that are defined for $x \in S$, satisfy

$$f(x) = af(3x) + bf(3x-1) + cf(3x-2),$$

if $0 \leq x \leq 1$ and are zero otherwise.

6. You are at a point (a, b) and need to reach another point (c, d). Both points are below the line $x = y$ and have integer coordinates. You can move in steps of length 1, either upwards or to the right, but you may not move to a point on the line $x = y$. How many different paths are there?

Austrian-Polish Mathematics Competition 1998

1. Let x_1, x_2, y_1, y_2 be real numbers such that $x_1^2 + x_2^2 \leq 1$. Prove the inequality

$$(x_1y_1 + x_2y_2 - 1)^2 \geq (x_1^2 + x_2^2 - 1)(y_1^2 + y_2^2 - 1).$$

2. Consider n points $P_1, P_2, \dots, P_n$ lying in that order on a straight line. We colour each point in white, red, green, blue or violet. A colouring is admissible if for each two consecutive points $P_i, P_{i+1} (i = 1, 2, \dots, n-1)$ either both points have the same colour, or at least one of them is white. How many admissible colourings are there?

3. Find all pairs of real numbers (x, y) satisfying the equations

$$2 - x^3 = y, \qquad 2 - y^3 = x.$$

4. Let m, n be positive integers. Prove that

$$\sum_{k=1}^{n} \left\lfloor \sqrt[k^2]{k^m} \right\rfloor \leq n + m(2^{m/4} - 1).$$

5. Find all pairs (a, b) of positive integers such that the equation

$$x^3 - 17x^2 + ax - b^2 = 0$$

has three integer roots (not necessarily distinct).

6. Distinct points A, B, C, D, E, F lie on a circle in that order. The tangents to the circle at the points A and D, and the lines BF and CE are concurrent. Prove that the lines AD, BC, EF are either parallel or concurrent.

7. Consider all pairs (a, b) of natural numbers such that the product $a^a b^b$ when written in base 10, ends with exactly 98 zeroes. Find the pair (a, b) for which the product ab is smallest.

8. Let $n > 2$ be a given natural number. In each unit square of an infinite grid is written a natural number. A polygon is admissible if it has area n and its sides lie on the grid lines. The sum of the numbers written in the squares contained in an admissible polygon is called the value of the polygon. Prove that if the values of any two congruent admissible polygons are equal, then all of the numbers written in the squares of the grid are equal.

9. Let K, L, M be the midpoints of sides BC, CA, AB, respectively, of triangle ABC. The points A, B, C divide the circumcircle of ABC into three arcs AB, BC, CA. Let X be the midpoint of the arc BC not containing A, let Y be the midpoint of the arc CA not containing B and let Z be the midpoint of the arc AB not containing C. Let R be the circumradius and r be the inradius of ABC. Prove that

$$r + KX + LY + MZ = 2R.$$

49th Romanian Mathematical Olympiad 1998

Selected problems from the final round

1. (7th form) Let n be a positive integer and $x_1, x_2, \ldots, x_n$ be integers such that

$$x_1^2 + x_2^2 + \cdots + x_n^2 + n^3 \leq (2n-1)(x_1 + x_2 + \cdots + x_n) + n^2.$$

Show that

(a) $x_i \geq 0$ for $i = 1, 2, \ldots, n$.

(b) $x_1 + x_2 + \ldots + x_n + n + 1$ is not a perfect square.

2. (7th form) Show that there is no positive integer n such that $n + k^2$ is a perfect square for at least n positive integer values of k.

3. (7th form) In the exterior of the triangle ABC with $\angle B > 45°$, $\angle C > 45°$, one constructs the right isosceles triangles ACM and ABN such that $\angle CAM = \angle BAN = 90°$ and, in the interior of ABC, the right isosceles triangle BCP with $\angle P = 90°$. Show that MNP is a right isosceles triangle.

4. (9th Form) Find integers a, b, c such that the polynomial

$$f(x) = ax^2 + bx + c$$

satisfies the equalities:

$$f(f(1)) = f(f(2)) = f(f(3)).$$

5. (9th Form) Let $ABCD$ be a cyclic quadrilateral. Prove that

$$|AC - BD| \leq |AB - CD|.$$

When does equality hold?

6. (10th Form) Let $n \geq 2$ be an integer and $M = \{1, 2, \ldots, n\}$. For every $k \in \{1, 2, \ldots, n-1\}$, let

$$x_k = \frac{1}{n+1} \sum_{\substack{A \subset M \\ |A| = k}} (\min A + \max A).$$

Prove that $x_1, x_2, \ldots, x_{n-1}$ are integers, not all divisible by 4.

41st International Mathematical Olympiad

Taejon, Korea, July 2000.

1. Two circles Γ_1 and Γ_2 intersect at M and N. Let ℓ be the common tangent to Γ_1 and Γ_2 so that M is closer to ℓ than N is. Let ℓ touch Γ_1 at A and Γ_2 at B. Let the line through M parallel to ℓ meet the circle Γ_1 again C and the circle Γ_2 again at D. Lines CA and DB meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

2. Let a, b, c be positive real numbers such that $abc = 1$. Prove that

$$\left(a - 1 + \frac{1}{b}\right)\left(b - 1 + \frac{1}{c}\right)\left(c - 1 + \frac{1}{a}\right) \leq 1.$$

3. Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point. For a positive real number λ, define a *move* as follows:

> choose any two fleas, at points A and B, with A to the left of B; let the flea at A jump to the point C on the line to the right of B with $BC/AB = \lambda$.

Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to the right of M.

4. A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one and a blue one, so that each box contains at least one card. A member of the audience selects two of the three boxes, chooses one card from each and announces the sum of the numbers on the chosen cards. Given this sum, the magician identifies the box from which no card has been chosen.

How many ways are there to put all the cards into the boxes so that this trick always works? (Two ways are considered different if at least one card is put into a different box.)

5. Determine whether or not there exists n such that

> n is divisible by exactly 200 different prime numbers and $2^n + 1$ is divisible by n.

6. Let AH_1, BH_2, CH_3 be the altitudes of an acute-angled triangle ABC. The incircle of the triangle ABC touches the sides BC, CA, AB at T_1, T_2, T_3, respectively. Let the lines ℓ_1, ℓ_2, ℓ_3 be the reflections of the lines H_2H_3, H_3H_1, H_1H_2 in the lines T_2T_3, T_3T_1, T_1T_2, respectively.

Prove that ℓ_1, ℓ_2, ℓ_3 determine a triangle whose vertices lie on the incircle of the triangle ABC.

• Solutions •

12th Nordic Mathematical Contest April 1998

1. Find all functions f from the rational numbers to the rational numbers satisfying

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all rational x and y.

Solution received from Jason Ying Hwei Ming (Victoria Junior College), Tan Chee Hau, Kwa Chin Lum, Christopher Tan Jun-yuan and Kiah Han Mao (all from Raffles Junior College). We present solutions by Jason, Chee Hau and Chin Lum. We shall prove by induction that $f(nr)=n^2f(r)$ for each positive integer n and each rational number r. The statement holds trivially for $n=1$. Assume that it holds for $n=1,2\ldots,m$. We have $f(mr+r)+f(mr-r)=2f(mr)+2f(r)$ which implies

$$\begin{aligned} f((m+1)r) &= -f((m-1)r)+2f(mr)+2f(r) \\ &= -(m-1)^2f(r)+2m^2f(r)+2f(r). \\ &= (m+1)^2f(r) \end{aligned}$$

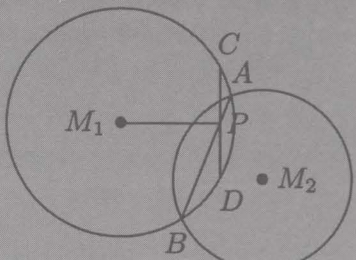
Thus the statement holds for $n=m+1$ as well and so the proof is complete by induction. For any rational number $r=p/q$ where p,q are coprime integers with $q>0$, we have

$$q^2f(r)=f(qr)=f(p)=p^2f(1).$$

Thus $f(r)=kr^2$ where $k=f(1)$ is a constant. Indeed, this function also satisfies the given conditions.

2. Let C_1 and C_2 be two circles which intersect at points A and B. Let M_1 be the centre of C_1 and M_2 the centre of C_2. Let P be a point on the line segment AB distinct from A and B so that $|AP|\neq|BP|$. Draw the line through P perpendicular to M_1P and denote by C and D its intersections with C_1 (see figure). Similarly (not drawn in the figure), draw the line through P perpendicular to M_2P and denote by E and F its intersections with C_2. Prove that C, D, E and F are the corners of a rectangle.

Similar solutions by Jason Ying Hwei Ming, Lim Yin (both from Victoria Junior College), Kiah Han Mao, Tan Chee Hau and Kwa



Chin Lum (all from Raffles Junior College). Since P is the foot of the perpendicular from M_1 to the chord CD, P bisects CD. Similarly, P bisects EF. Since P bisects the two diagonals, $CFDE$ is a parallelogram. Next since $CP \cdot PD = AP \cdot PB = EP \cdot PF$, we have $CP = PF$. Thus the diagonals are equal and consequently $CFDE$ is a rectangle.

3. (a) For which positive integer n does there exist a sequence $x_1, \ldots, x_n$ containing each of the numbers $1, 2, \ldots, n$ exactly once and such that k divides $x_1 + x_2 + \cdots + x_k$ for $k = 1, 2, \ldots, n$?

(b) Does there exist an infinite sequence $x_1, x_2, \ldots$ containing every positive integer exactly once and such that for any positive integer k, k divides $x_1 + x_2 + \cdots + x_k$?

Tan Chee Hau and Kwa Chin Lum (both of Raffles Junior College) submitted similar solutions to part (a). Let $S_k = x_1 + \cdots + x_k$. Since $n \mid S_n = n(n+1)/2$, we conclude that n must be odd. Now assume that n is odd. Then

$$S_{n-1} = \frac{n(n+1)}{2} - x_n \equiv 0 \pmod{n-1}.$$

Thus $x_n \equiv (n+1)/2 \pmod{n-1}$, whence $x_n = (n+1)/2$ because $(n+1)/2 + (n-1) > n$. Next, we have

$$\begin{aligned} S_{n-2} + x_{n-1} = S_{n-1} &= S_n - x_n \\ &= (n-1)(n+1)/2 \equiv (n+1)/2 \pmod{n-2}. \end{aligned}$$

This implies that $x_{n-1} \equiv (n+1)/2 \pmod{n-2}$. Since $(n+1)/2 + (n-2) > n$ when $n > 3$, we have $x_{n-1} = (n+1)/2 = x_n$ if $n > 3$. Thus no such sequence can exist if $n > 3$.

For $n = 3$, $x_1 = 1, x_2 = 3, x_3 = 2$ and for $n = 1$, $x_1 = 1$ are the only sequences with this property.

Solution to (b) by Kwa Chin Lum (Raffles Junior College). Let $x_1 = 1$. Then $1 \mid x_1$. Suppose a sequence $x_1, \ldots, x_k$, has been constructed so that $i \mid x_1 + \cdots + x_i$, $i = 1, \ldots, k$. Then $x_1 + \cdots + x_k = mk$ for some positive integer m. Let p be the smallest positive integer such that $x_i \neq p$ for $i = 1, \ldots, k$. If there exists a positive integer n such that $m + n(k+1) = p$, then let $x_{k+1} = p$ and we have a sequence such that $i \mid x_1 + \cdots + x_i$, $i = 1, \ldots, k+1$. If not then one can choose n_0 so that

$$\begin{aligned} &(m + n_0)(k+1) > x_i,\ i = 1, \ldots, k, \\ \text{and} \quad &(m + n_0)(k+1) \equiv -p \pmod{k+2}. \end{aligned}$$

This is always possible by choosing a large n_0 so that $n_0 \equiv p - m \pmod{k+2}$. Then letting $x_{k+1} = (m + n_0)(k+1)$ and $x_{k+2} = p$

will give a sequence such that $i \mid x_1 + \cdots + x_i$, $i = 1, \ldots, k+2$. In each case, we can extend the sequence to include p. Thus the required sequence exists.

4. Let n be a positive integer. Count the number of $k \in \{0, 1, \ldots, n\}$ for which $\binom{n}{k}$ is odd. Prove that this number is a power of 2, i.e., of the form 2^p for some non-negative integer p.

We'll present two solutions. The first is by Kiah Han Mao and Tan Chee Hau (both of Raffles Junior College). Since $\binom{n}{k} = n!/k!(n-k)!$, for it to be odd, we must have $\alpha_n = \alpha_k + \alpha_{n-k}$, where α_m denotes the highest power 2 that divides the integer m. Consider the binary representation of $k, n-k, n$:

$$k = \overline{x_n x_{n-1} \ldots x_0}, \quad n-k = \overline{y_n y_{n-1} \ldots y_0}, \quad n = \overline{z_n z_{n-1} \ldots z_0}.$$

We shall prove that $\alpha_n = \alpha_k + \alpha_{n-k}$ if and only if $x_i + y_i = z_i$ for $i = 0, \ldots, n$. We have

$$\alpha_n = \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{2^2} \right\rfloor + \cdots = \overline{z_n z_{n-1} \ldots z_1} + \overline{z_n z_{n-1} \ldots z_2} + \cdots + \overline{z_n}.$$

If $x_i + y_i = z_i$ for $i = 0, \ldots, n$, then $\alpha_n = \alpha_k + \alpha_{n-k}$. If there exists i such that $x_i + y_i \neq z_i$, let m be the largest integer such that $x_m + y_m \neq z_m$. If $x_m + y_m > z_m$, then $x_m + y_m \geq 1$ and there must be a 'carry over' in the addition causing $x_{m+1} + y_{m+1} \neq z_{m+1}$. Thus $x_m + y_m < z_m$, i.e., $x_m = y_m = 0$ and $z_m = 1$. This implies that

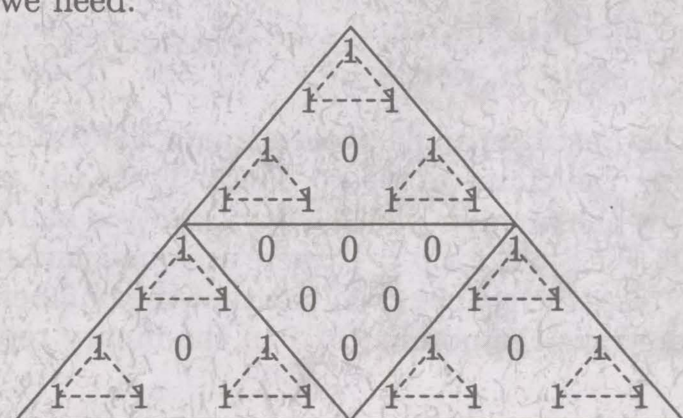
$$\overline{z_n z_{n-1} \ldots z_m} > \overline{x_n x_{n-1} \ldots x_m} + \overline{y_n y_{n-1} \ldots y_m}.$$

Since

$$\overline{z_n z_{n-1} \ldots z_i} \geq \overline{x_n x_{n-1} \ldots x_i} + \overline{y_n y_{n-1} \ldots y_i} \quad \text{for} \quad i = 1, 2, \ldots n,$$

we have $\alpha_n > \alpha_k + \alpha_{n-k}$. With this the proof of the assertion is complete. If $\alpha_n = \alpha_k + \alpha_{n-k}$, then $z_i = 0$ implies $x_i = y_i = 0$ and $z_i = 1$ implies exactly one of x_i, y_i is 1. Thus the number of terms $\binom{n}{k}$, $k = 0, \ldots, n$ which are odd is 2^{r_n}, where r_n is the number of ones in the binary representation of n.

The second solution is due to Kwa Chin Lum and Christopher Tan Junyuan (both from Raffles Junior College). It is easy to notice that in the Pascal triangle when the entries are taken mod 2, certain triangles drawn repeat themselves (see figure below). (The repeating pattern of triangles is known as Serpinski's casket.) To prove this we need:



(1) $\binom{2^p-1}{k} \equiv 1 \pmod 2$ for $k = 1, \ldots, 2^p - 1$ since

$$\binom{2^p-1}{k} = \frac{2^p-1}{1} \cdot \frac{2^p-2}{2} \cdots \frac{2^p-k}{k}$$

and each term of the product is a quotient of two odd numbers as

$$\frac{2^p-q}{q} = \frac{2^p-2^r s}{2^r s} = \frac{2^{p-r}-s}{s}, \quad \text{where } q = 2^r s, \text{ and } s \text{ is odd.}$$

(2) Let $n = 2^p - 1$. Then from (1)

$$\binom{n+1}{k} \equiv \begin{cases} 0 \pmod 2 & \text{if } k = 1, 2, \ldots, n \\ 1 \pmod 2 & \text{if } k = 0, n+1. \end{cases}$$

(3) For any nonnegative integer p and any integer q, let $\binom{p}{q} = 0$ if $q < 0$ or $q > p$. Then it is easy to see that $\binom{n+1+k}{i} = \binom{k}{0}\binom{n+1}{i-k} + \binom{k}{1}\binom{n+1}{i-k+1} + \cdots + \binom{k}{k}\binom{n+1}{i}$. This follows readily from the recursive formula

$$\begin{aligned}\binom{p+1}{i} &= \binom{p}{i-1} + \binom{p}{i} \\ &= \binom{2}{0}\binom{p-1}{i-2} + \binom{2}{1}\binom{p-1}{i-1} + \binom{2}{2}\binom{p-1}{i} = \cdots\end{aligned}$$

which holds true for the generalized binomial coefficients as well.

(4) Thus for each $k = 0, 1, \ldots, n$ and $i = 0, 1, \ldots, k$, we have

$$\begin{aligned}\binom{n+1+k}{n+1+k-i} &= \binom{n+1+k}{i} \\ &\equiv \binom{k}{k-i}\binom{n+1}{0} \equiv \binom{k}{i} \pmod 2.\end{aligned}$$

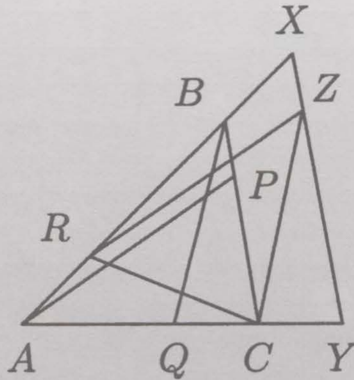
Also, for each $k = 0, 1, \ldots, n$ and $i = k+1, \ldots, n$, since $1 \le i-k < i-k+1 < \cdots < i \le n$, we have $\binom{n+1+k}{i} \equiv 0 \pmod 2$.

(5) In the Pascal triangle, we designate the top row as row 0 and increasing the count as we go down. From (3) and (4), one concludes that the number of 1 in row $i (< 2^p)$ is half the number of 1 in row $2^p + i$. Since for $p = 1$, there are one 1 and two 1 in row 0 and row 1, respectively, we conclude that the number of 1 in row i is 2^{r_i}, where r_i is the number of 1 in the binary representation of i.

First Japan Mathematical Olympiad, 1991

Final Round

1. On a triangle ABC, let P, Q, R be the points which divide the segments BC, CA, AB, respectively, in the ratio $t : 1-t$. Let K be the area of the triangle whose three edges have the same length with the segments AP, BQ and CR and let L be the area of triangle ABC. Find K/L in terms of t.

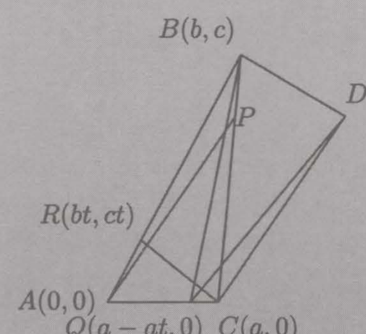


Five solutions were received: from Kwa Chin Lum and Kiah Han Mao (Raffles Junior College), Lim Yin and Jason Ying Hwei Ming (Victoria Junior College), as well as A. R. Pargeter (England). We first present Lim Yin's solution. We have $AR/RB = BP/PC = CQ/QA = t/(1-t)$. Extend AB to X and AC to Y so that $BX = AR$ and $QC = CY$. Then $\triangle ABC$ is similar to $\triangle AXY$ since $BX/AB = t = CY/AC$. So $XY \parallel BC$ and $XY/BC = 1+t$. Now let Z be the point on XY such that $XZ = BP$. Then $\triangle RXZ$ and $\triangle ABP$ are congruent and so $AP = RZ$. From $XY/BC = 1+t$ and $XZ/BC = BP/BC = t$, we have $ZY/BC = (1+t) - t = 1$. Thus $ZY = BC$ and consequently, $\triangle BCQ$ is congruent to $\triangle ZYC$ and $BQ = ZC$. Thus $\triangle RZC$ is the triangle whose sides are equal to AP, BP, RC. For any $\triangle ABC$ let (ABC) denote its area. Then $(AXY) = (1+t)^2(ABC) = (1+t)^2L$. $(ARC) = (RXZ) = (ZYC) = tL$. So

$$\begin{aligned} K = (RZC) &= (AXY) - (ARC) - (RXZ) - (ZYC) \\ &= (1+t)^2L - 3tL \end{aligned}$$

and $K/L = t^2 - t + 1$.

Jason's solution. Set up the coordinates with $A = (0,0)$, $B = (b,c)$, $C = (a,0)$. Then we have $R = (bt, ct)$, $Q = (a-at, 0)$ and $P = (at + b - bt, c - ct)$. Let D be the point so that $ACDB$ is a parallelogram. Then



$$\begin{aligned} \overline{AD} &= \overline{AC} + \overline{CD} = \overline{AC} + \overline{RB} \\ &= (a,0) + (b,c) - (bt,ct) = (a+b-bt, c-ct). \end{aligned}$$

Thus $D = (a+b-bt, c-ct)$ and $\overline{QP} = (at+b-bt, c-ct) = \overline{AP}$. Therefore $\triangle BQP$ has its sides equal to the segments AP, BQ, CR. Hence

$$\frac{K}{L} = \frac{(BDQ)}{(ABC)} = \frac{\begin{vmatrix} b & c & 1 \\ a-at & 0 & 1 \\ a+b-bt & c-ct & 1 \end{vmatrix}}{\begin{vmatrix} b & c & 1 \\ 0 & 0 & 1 \\ a & 0 & 1 \end{vmatrix}} = t^2 - t + 1.$$

Han Mao's solution. Let $\overline{AB} = \mathbf{b}$ and $\overline{AC} = \mathbf{c}$. Then $(ABC) = \frac{1}{2}|\mathbf{b} \times \mathbf{c}|$. We also have $\overline{BQ} = \overline{BA} + \overline{AQ} = -\mathbf{b} + t\mathbf{c}$, $\overline{CR} = \overline{CA} + \overline{AR} = -\mathbf{c} + (1-t)\mathbf{b}$, $\overline{AP} = \overline{AC} + \overline{CP} = \mathbf{c} + t(\mathbf{b} - \mathbf{c})$. Thus $\overline{BQ} + \overline{CR} + \overline{AP} = \mathbf{0}$. Hence

$$\begin{aligned} K &= \frac{1}{2}|\overline{BQ} \times \overline{CR}| \\ &= \frac{1}{2}|t\mathbf{c} - \mathbf{b} \times (1-t)\mathbf{b} - \mathbf{c})| \\ &= \frac{(t^2 - t + 1)}{2}|\mathbf{b} \times \mathbf{c}| = (t^2 - t + 1)L \end{aligned}$$

2. Let $\mathbb{N}$ be the set of all positive integers. The maps $p, q : \mathbb{N} \to \mathbb{N}$ are defined as follows:

$$p(1) = 2, p(2) = 3, p(3) = 4, p(4) = 1; p(n) = n \text{ if } n \geq 5.$$

$$q(1) = 3, q(2) = 4, q(3) = 2, q(4) = 1; q(n) = n \text{ if } n \geq 5.$$

(a) There exists a map $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = p(n) + 2$ for all $n \in \mathbb{N}$. Find an example of such a map f.

(b) Show that there does not exist a map $g : \mathbb{N} \to \mathbb{N}$ such that $g(g(n)) = q(n) + 2$ for all $n \in \mathbb{N}$.

(a): Solution by Lim Yin (Victoria Junior College), Christopher Tan Junyuan, Tan Chee Hau and Kwa Chin Lum (all from Raffles Junior College). The following function has the required properties:

$$f(n) = \begin{cases} 2 & \text{if } n = 1 \\ 4 & \text{if } n = 2 \\ 7 & \text{if } n = 3 \\ 5 & \text{if } n = 4 \\ 3 & \text{if } n = 5 \\ n+3 & \text{if } n = 6, 8, \ldots \\ n-1 & \text{if } n = 7, 9, \ldots \end{cases}$$

Solution (b): Solution by Lim Yin (Victoria Junior College). Also solved by Christopher Tan Junyuan, Tan Chee Hau and Kwa Chin Lum (all from Raffles Junior College. Suppose such a function g exists. Then

$$g^4(n) = \begin{cases} 7 & \text{if } n = 1 \\ 8 & \text{if } n = 2 \\ 3 & \text{if } n = 3 \\ 4 & \text{if } n = 4 \\ n+4 & \text{if } n \geq 5 \end{cases}$$

Let $g(3) = k$. Then

$$g^4(k) = g^4(g(3)) = g^5(3) = g(g^4(3)) = g(3) = k.$$

Thus $k = 3$ or 4. If $g(3) = 3$, then $g(g(3)) = g(3) = 3$, a contradiction. If $g(3) = 4$, then $4 = g(g(3)) = g(4)$ and $g(g(4)) = g(4) = 4$ again a contradiction. Thus no such function can exist.

3. Let A be a positive integer of 16 digits in the decimal system. Prove that we can chose some successive digits from A such that their product is a square of an integer.

Similar solutions by Lim Yin (Victoria Junior College), Kwa Chin Lum and Tan Chee Hau (both from Raffles Junior College). Let $B(i, j)$ represent the product of the digits of A from the ith digit to the jth digit, $i \leq j$. The possible prime factors are $2, 3, 5, 7$. Consider the numbers $B(1, 1), B(1, 2), \ldots, B(1, 16)$. We can write $B(1, i) = 2^{p_i}3^{q_i}5^{r_i}7^{s_i}$. If there is an i, such that p_i, q_i, r_i, s_i are all even, then $B(1, i)$ is a perfect square. If not, then among the 16 4-tuples $(p_i, q_i, r_i, s_i), i = 1, 2 \ldots, 16$, two must have the same parity, say (p_m, q_m, r_m, s_m) and (p_n, q_n, r_n, s_n), $m < n$ have the same parity, i.e., $p_n - p_m, q_n - q_m, r_n - r_m, s_n - s_m$ are all even. Then $B(m + 1, n) = B(1, n)/B(1, m)$ is a perfect square.

4. On a rectangle chess board of size 10×14, the squares are coloured white and black alternately. We write 0 or 1 in every square so that every row and every column contains an odd number of 1. Prove that the total number of 1 in the black squares is even.

Similar solutions by Lim Yin (Victoria Junior College), Kwa Chin Lum and Tan Chee Hau (both of Raffles Junior College) and Lim Chong Jie. Also solved by Kiah Han Mao (Raffles Junior College). Let the $(1, 1)$ square be black. Add up the odd columns and the even rows. This is the sum of 12 odd numbers so it is even. Now each black cell appears exactly once in the sum and each white cell appears either twice or none at all in the sum. Since the sum is even, the sum of the numbers in the black cells is even, or there is an even number of ones in the black cells.

(*Note: A similar version of this problem appeared as Problem 1 of Ukrainian Mathematical Olympiad 1997. The second solution published in Medley (Vol. 26, No. 2, December 1999) works for this case and is in fact the solution presented here.*)

5. Let A be a set of n points on a plane ($n \geq 2$). Prove that there exists a closed circular disk with two points of A at the two ends of a diameter and which contains at least $\lfloor n/3 \rfloor$ points of A. (*Note: For any real number x, $\lfloor x \rfloor$ denotes the largest integer $\leq x$.*)

No solution was received. We present the official solution. Let D be the smallest closed disk which contains all points of A. We denote the boundary of D by ∂D. If there are only two points of A on ∂D, then these two points are on a diameter of D and D is the required disk. If there are at least three points of A on ∂D, then we can find three points P, Q, R such that ΔPQR is either acute-angled or right-angled. If ΔPQR is right-angled, then D is the required disk. If ΔPQR is acute-angled, let D_1, D_2, D_3 be closed disks whose diameter are PQ, QR, RP, respectively. It is easy to see that one of the D_i's is the required disk.

Georgian Mathematical Olympiad, May 1997

Selected problems from the final round

1. (9th Form) Prove that for any positive integer n, the following equalities hold:

$$\lfloor\sqrt{n}+\sqrt{n+1}\rfloor = \lfloor\sqrt{4n+1}\rfloor = \lfloor\sqrt{4n+2}\rfloor = \lfloor\sqrt{4n+3}\rfloor.$$

Solution by Tan Chee Hau, Kiah Han Mao, Kwa Chin Lum (all from Raffles Junior College) and Lim Chong Jie. First, we have

$$\begin{aligned} & 4n^2+4n+1 > 4n^2+4n > 4n^2 \\ \Rightarrow\quad & 2n+1 > 2\sqrt{n(n+1)} > 2n \\ \Rightarrow\quad & 4n+2 > n+2\sqrt{n(n+1)}+(n+1) > 4n+1 \\ \Rightarrow\quad & \sqrt{4n+2} > \sqrt{n}+\sqrt{n+1} > \sqrt{4n+1}. \end{aligned}$$

Thus

$$\lfloor\sqrt{4n+2}\rfloor \geq \lfloor\sqrt{n}+\sqrt{n+1}\rfloor \geq \lfloor\sqrt{4n+1}\rfloor.$$

If there is an integer k such that $\sqrt{4n+1} < k \leq \sqrt{4n+3}$, then $4n+1 < k^2 \leq 4n+3$, i.e., $k^2 \equiv 2, 3 \pmod 4$ which is impossible. Thus we conclude that $\lfloor\sqrt{4n+1}\rfloor = \lfloor\sqrt{4n+3}\rfloor$. From this the conclusion follows.

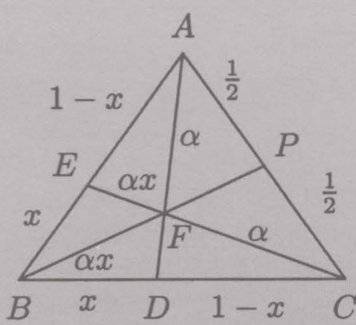
2. (9th Form) There are 40 participants in a mathematical competition. Each problem was marked with a +, a − or 0. After all the papers were marked it was found that no two papers had the same number of + and the same number of − marks simultaneously. What was the smallest number of problems that could have been offered to the contestants?

Solution by Tan Chee Hau, Kwa Chin Lum, Kiah Han Mao (all from Raffles Junior College) and Lim Chong Jie. Suppose there are n questions. If '0' is not assigned to any question, then the number of questions assigned '+' is i, while the number of questions assigned '−' is $n-i$, $i = 0, \ldots, n$. If '0' is assigned to one question, then the number of questions assigned '+' is i, while the number of questions assigned '−' is $n-1-i$, $i = 0, \ldots, n-1$, etc. Thus the total number of ways of assigning the number of '+', '−' and '0' is $1+2+\cdots+(n+1) = (n+1)(n+2)/2$. Since there are

40 students, we have $(n+1)(n+2)/2 \geq 40$, which means $n \geq 8$. Thus the smallest number of questions is 8. It is clear from the forgoing discussion that with 8 questions, the required assignment is possible,

3. (9th Form) In the equilateral triangle ABC, points D and E are chosen on the sides BC and BA, respectively, so that $\angle DAC = \angle ECA$. The lines AD and CE meet at a point F. The incircles of the triangle AFC and the quadrilateral $BDFE$ have equal radii. Find the radius of these circles if the length of the side of ABC is a.

We present similar solutions by Tan Chee Hau, Kwa Chin Lum, Kiah Han Mao(all from Raffles Junior College). Also solved by A. R. Pargeter (England). Let the equilateral triangle be of unit side. Then the actual answer would be a times the answer obtained in this case. Let the radius of the two incircles be r and let $BD = BE = x$ (see Figure). Apply Menelaus' Theorem to $\triangle ABD$ and $\triangle ABP$, we have



$$\frac{1-x}{x} \cdot \frac{1}{1-x} \cdot \frac{DF}{FA} = 1, \quad \text{i.e.,} \quad \frac{DF}{FA} = \frac{EF}{FC} = x$$
$$\frac{1-x}{x} \cdot \frac{BF}{FP} \cdot \frac{1/2}{1} = 1, \quad \text{i.e.,} \quad \frac{BF}{FP} = \frac{2x}{1-x}.$$

The latter also implies that $FP = \frac{\sqrt{3}(1-x)}{2(1+x)}$. Let $AF = \alpha$, then we have $DF = \alpha x = EF$ and $FC = \alpha$. For each polygon $AB \ldots D$, we denote its area by $(AB \ldots D)$. Let $(BEF) = Q$, $(AFE) = P$, $(AFP) = R$, we have

$$\frac{P}{Q} = \frac{1-x}{x}, \quad \text{and} \quad \frac{P+Q}{R} = \frac{1+(P/Q)}{(R/Q)} = \frac{BF}{FP} = \frac{2x}{1-x}.$$

Thus $\frac{Q}{R} = \frac{2x^2}{1-x}$. We also have

$$2(AFC) = r(1+2\alpha) \quad \text{and} \quad 2(BEFD) = r(2x + 2\alpha x).$$

Since $(BEFD)/(AFC) = Q/R$, we have

$$\frac{x}{1-x} = \frac{1+\alpha}{1+2\alpha}, \quad \text{which implies} \quad \alpha = \frac{1-2x}{3x-1}.$$

Consider the right-angled triangle APF, we have $AP^2 + FP^2 = \alpha^2$, or

$$\frac{1}{4} + \frac{3}{4}\left(\frac{1-x}{1+x}\right)^2 = \left(\frac{1-2x}{3x-1}\right)^2$$
$$\Leftrightarrow \quad x(x-1)(5x^2 - 14x + 5) = 0.$$

Since $0 < x < 1$, we have $x = (7 - 2\sqrt{6})/5$. We are now ready to calculate r. Since (AFC) is the product of its semiperimeter and its inradius and also $(AFC) = \frac{FP}{BP}(ABC)$, we have:

$$\frac{r(1+2\alpha)}{2} = \frac{1}{2}\frac{\sqrt{3}}{2}\left(\frac{1-x}{1+x}\right).$$

Thus

$$r = \frac{\sqrt{3}}{2}\left(\frac{1-x}{1+x}\right)\left(\frac{1}{1+2\alpha}\right) = \frac{\sqrt{3}-\sqrt{2}}{2}$$

4. (10th Form) Find all triples (x, y, z) of integers satisfying the inequality:

$$x^2 + y^2 + z^2 < xy + 3y + 2z.$$

Solution by Tan Chee Hau, Kwa Chin Lum, Kiah Han Mao (all from Raffles Junior College). By completing squares, we have

$$\begin{aligned} & x^2 + y^2 + z^2 < xy + 3y + 2z \\ \Leftrightarrow \quad & (2x - y)^2 + (3y^2 + 4z^2 - 12y - 8z) < 0 \end{aligned}$$

Thus for a solution to exist,

$$\begin{aligned} 3y^2 + 4z^2 - 12y - 8z &= 3(y-2)^2 + 4(z-3)(z+1) \\ &= 4(z-1)^2 + (3y^2 - 12y - 4) \\ &< 0. \end{aligned}$$

Thus we need

$$(z-3)(z+1) < 0 \quad \text{and} \quad 3y^2 - 12y - 4 < 0,$$

i.e.,

$$z = 0, 1, 2 \quad \text{and} \quad y = 0, 1, 2, 3, 4.$$

The solutions can now be worked out by considering all the cases. For example, when $(y, z) = (0, 0)$, $x^2 < 0$ and there is no solution; when $(y, z) = (0, 1)$, $x^2 < 1$ and $x = 0$ is the only solution. All the solutions are:

$$\begin{aligned} (x, y, z) = {} & (0,0,1), (0,1,0), (1,1,0), (-1,1,1), (0,1,1), (1,1,1), \\ & (2,1,1), (0,1,2), (1,1,2), (0,2,0), (1,2,0), (2,2,0), \\ & (0,2,1), (1,2,1), (2,2,1), (0,2,2), (1,2,2), (2,2,2), \\ & (1,3,0), (2,3,0), (0,3,1), (1,3,1), (2,3,1), (3,3,1), \\ & (1,3,2), (2,3,2), (2,4,1). \end{aligned}$$

5. (10th Form) Determine whether or not it is possible to fill an $n \times n$ table with entries equal to $1, -1$ or 0 so that when calculating the sums of the entries along the rows and the columns one could get 20 different numbers.

Solution by Tan Chee Hau, Kiah Han Mao, Kwa Chin Lum (all from Raffles Junior College). In order to have 20 different row and column sums, $n \geq 10$. The following table gives a solution for $n = 10$.

1	1	1	1	1	1	1	1	0	0
1	1	1	1	1	1	1	0	0	−1
1	1	1	1	1	1	0	0	−1	−1
1	1	1	1	1	0	0	−1	−1	−1
1	1	1	1	1	0	−1	−1	−1	−1
1	1	1	1	1	−1	−1	−1	−1	−1
1	1	1	1	0	−1	−1	−1	−1	−1
1	1	1	0	−1	−1	−1	−1	−1	−1
1	1	0	−1	−1	−1	−1	−1	−1	−1
1	0	−1	−1	−1	−1	−1	−1	−1	−1

For larger values of n, simply put this table anywhere, say the top left corner and enter 0 at the other places.

6. (10th Form) Prove that in any triangle, $pR \geq 2S$, where p, R, S are, respectively, the semiperimeter, circumradius and the area of the triangle.

Solution by Tan Chee Hau, Kwa Chin Lum (both from Raffles Junior College). Also solved by A. R. Pargeter (England). Let O and I be the circumcentre and the incentre. Then by Euler's Theorem,

$$OI^2 = R^2 - 2rR = R(R - 2r) \geq 0.$$

Thus $R \geq 2r$ and $pR \geq 2rP = 2S$ as required.

7. (11th Form) Two positive numbers are written on a board. At each step you must perform one of the following:

(i) choose one of the numbers, say a, on the board and write down either a^2 or $1/a$.

(ii) choose two numbers, say a, b, on the board and write down either $a + b$ or $|a - b|$. How should you proceed in order that the product of the two initial numbers will eventually be written on the board?

Solution by Tan Chee Hau, Kwa Chin Lum(both from Raffles Junior College). Let $a, b (a > b)$ be the initial numbers. In each of the following steps write:

(1) $\frac{1}{a}, \frac{1}{b}$

(2) $\dfrac{1}{a}+\dfrac{1}{b}=\dfrac{a+b}{ab},\quad \dfrac{1}{b}-\dfrac{1}{a}=\dfrac{a-b}{ab}$

(3) $\left(\dfrac{a+b}{ab}\right)^2=\dfrac{a^2+2ab+b^2}{a^2b^2},\quad \left(\dfrac{a-b}{ab}\right)^2=\dfrac{a^2-2ab+b^2}{a^2b^2}$

(4) $\dfrac{a^2+2ab+b^2}{a^2b^2}-\dfrac{a^2-2ab+b^2}{a^2b^2}=\dfrac{4ab}{a^2b^2}=\dfrac{4}{ab}$

(5) $\dfrac{1}{4/ab}=\dfrac{ab}{4}$ twice

(6) write $\frac{ab}{4}+\frac{ab}{4}=\frac{ab}{2}$ twice

(7) $\dfrac{ab}{2}+\dfrac{ab}{2}=ab$

40th International Mathematical Olympiad

Bucharest, Romania, July 1999

1. Determine all finite sets S of at least three points in the plane which satisfy the following condition:

> for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

Solution. Let G be the centre of gravity of the set S. Since the perpendicular bisector of the line joining every pair of points is an axis of symmetry, G lies on the perpendicular bisector. Thus the perpendicular bisectors meet at a common point G. Thus the points lie on a circle. Let a, b, c be three consecutive points. Since the perpendicular of ac is an axis of symmetry, b must lie on it. Thus the lengths of ab and bc are equal. Thus the points are the vertices of a regular polygon.

Alternative solution: One can show easily that the boundary of the convex hull of the points form a regular polygon as in the second half of the previous proof. The only thing left to do is to prove that there is no point in the interior. This can be proved by contradiction.

2. Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1\leq i<j\leq n} x_i x_j (x_i^2+x_j^2) \leq C\Big(\sum_{1\leq i\leq n} x_i\Big)^4$$

holds for all real numbers $x_1, \ldots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

Solution: The inequality is symmetric and homogeneous, so we can assume $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$ and $\sum x_i = 1$. In this case we have to maximize the sum

$$F(x_1, \ldots, x_n) = \sum_{i<j} x_i x_j (x_i^2 + x_j^2).$$

Let x_{k+1} be the last nonzero coordinate and we assume that $k \geq 2$. We shall replace $x = (x_1, \ldots, x_k, x_{k+1}, 0, \ldots, 0)$ with

$$x' = (x_1, \ldots, x_{k-1}, x_k + x_{k+1}, 0, \ldots, 0)$$

to increase the value of F as shown below:

$$\begin{aligned} F(x') - F(x) &= x_k x_{k+1} [3(x_k + x_{k+1}) \sum_{i=1}^{k-1} x_i - x_k^2 - x_{k+1}^2] \\ &= x_k x_{k+1} [3(x_k + x_{k+1})(1 - x_k - x_{k+1}) - x_k^2 - x_{k+1}^2] \\ &= x_k x_{k+1} (x_k + x_{k+1})(3 - 4(x_k + x_{k+1})) + 2x_k x_{k+1}]. \end{aligned}$$

From

$$1 \geq x_1 + x_k + x_{k+1} \geq \frac{1}{2}(x_k + x_{k+1}) + x_k + x_{k+1}$$

it follows that $2/3 \geq x_k + x_{k+1}$, and therefore $F(x') - F(x) > 0$. After several such substitutions, we have

$$\begin{aligned} F(x) &\leq F(a, b, 0, \ldots, 0) = ab(a^2 + b^2) \\ &= \frac{1}{2}(2ab)(1 - 2ab) \leq \frac{1}{8} = F(1/2, 1/2, 0, \ldots, 0). \end{aligned}$$

Thus $C = 1/8$ and equality occurs if and only if two of the x_i's are equal (possibly zero) and the remaining variables are zero.

3. Consider an $n \times n$ square board, where n is a fixed positive even integer. The board is divided into n^2 unit squares. We say that two different squares on the board are *adjacent* if they have a common side. N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square. Determine the smallest possible value of N.

Solution. Let $n = 2k$. First colour the board black and white like a chessboard. We say that a set S of cells is *covered* by a set M of marked cells if every cell in S is neigbour to a cell in M. Let $f_w(n)$ be the minimal number of white cells that must be marked so as to cover all the black cells. Define similarly $f_b(n)$. Due to symmetry of the chessboard (n is even), we have

$$f_w(n) = f_b(n).$$

Since black cells can only be covered by marked white cells and vice versa, we have

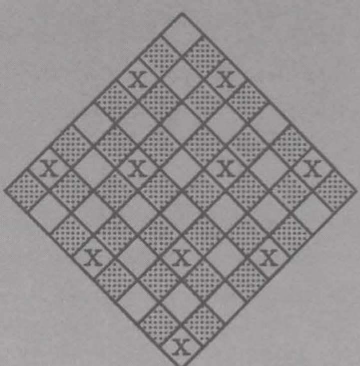
$$N = f_w(n) + f_b(n).$$

Place the board so that the longest black diagonal is horizontal. Now consider the horizontal rows of white cells. Mark the odd cells of the first row, third row, fifth row, etc. (The figure illustrates the case $n = 8$). Call this set of cells M. The number of cells in M is

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

It is easy to see that the black cells are covered by M. Thus

$$f_w(n) \leq \frac{k(k+1)}{2}$$



Now consider the cells in M. No two of them have a common black neighbour. So we need to mark at least $k(k+1)/2$ black cells in order to cover M. Therefore

$$f_b(n) \geq \frac{k(k+1)}{2}.$$

Hence

$$f_b(n) = f_w(n) = k(k+1)/2.$$

Thus

$$N = f_b(n) + f_w(n) = k(k+1).$$

4. Determine all pairs (p, q) of positive integers such that p is prime, $n \leq 2p$, and $(p-1)^n + 1$ is divisible by n^{p-1}.

Solution. Clearly $(1, p)$ and $(2, 2)$ are solutions and for other solutions we have $p \geq 3$. Now assume that $n \geq 2$ and $p \geq 3$. Since $(p-1)^n + 1$ is odd and is divisible by n^{p-1}, n must be odd. Thus $n < 2p$. Let q be the smallest prime divisor of n. From $q \mid (p-1)^n + 1$, we have

$$(p-1)^n \equiv -1 \pmod{q} \quad \text{and} \quad \gcd(q, p-1) = 1.$$

But $\gcd(n, q-1) = 1$ (from the choice of q), there exist integers u and v such that $un + v(q-1) = 1$, whence

$$p - 1 \equiv (p-1)^{un}(p-1)^{v(q-1)} \equiv (-1)^u 1^v \equiv -1 \pmod{q},$$

because u must be odd. This shows $q \mid p$ and therefore $q = p$. Hence $n = p$. Now

$$\begin{aligned} p^{p-1} &\mid (p-1)^p + 1 \\ &= p^2\left(p^{p-2} - \binom{p}{1}p^{p-3} + \cdots + \binom{p}{p-3}p - \binom{p}{p-2} + 1\right) \end{aligned}$$

Since every term in the bracket except the last is divisible by p, we have $p - 1 \leq 2$. Thus $p = 3 = n$. Indeed $(3,3)$ is a solution.

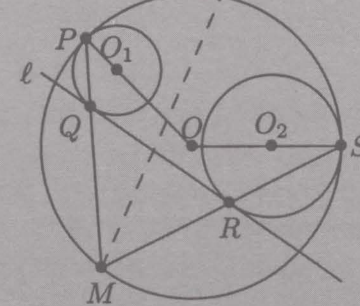
In conclusion, the only solutions are $(1,p),(2,2),(3,3)$.

5. Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the centre of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B. The lines MA and MB meet Γ_1 at C and D, respectively.

Prove that CD is tangent to Γ_2.

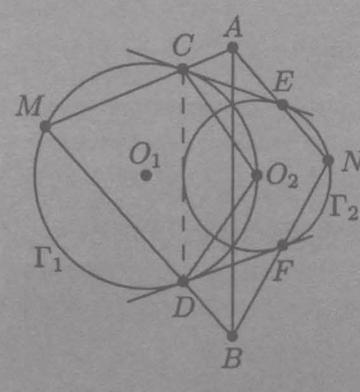
Solution: We first prove a lemma.

Lemma: Let Γ_1 and Γ_2 be two circles such that one does not contain the other. Let Γ be a circle containing both Γ_1, Γ_2 and touches Γ_1 at P, Γ_2 at S. Let ℓ be a chord of Γ which is tangent to Γ_1 at Q and Γ_2 at R, with both circles on the same side of the chord. Then PQ and SR meet at a point M which is on Γ. Furthermore M is on the radical axis of Γ_1 and Γ_2.



Proof: Let O, O_1, O_2 be the centres of $\Gamma, \Gamma_1, \Gamma_2$, respectively. Produce PQ to meet Γ at X. Then it is easy to see that $O_1Q \| OX$. Thus the tangent at X is parallel to ℓ. Produce SR to meet Γ at Y. A similar consideration shows that the tangent at Y is also parallel to ℓ. Thus $X = Y = M$. Since ℓ is parallel to the tangent at M, we have $\angle SPQ = 180° - \angle QRS$. Thus $PQRS$ is cyclic. Hence the powers of M with respect to Γ_1 and Γ_2 are equal and M is on the radical axis of both. This completes the proof of the lemma.

Let E be the intersection of NA with Γ_2 and F be the intersection of NB with Γ_2. From the lemma, and since the radical axis in this case is the common chord AB, we known that CE and DF are both common tangents of Γ_1 and Γ_2. Thus O_1O_2 is the perpendicular bisector of CD, i.e., $O_2C = O_2D$ and $\angle O_2CD = \angle O_2DC$. Since CE is tangent to Γ_1, $\angle ECO_2 = \angle O_2DC = \angle O_2CD$. Thus CD is tangent to Γ_2.



6. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

Solution. Let $A = \operatorname{Im} f$ and $c = f(0)$. By putting $x = y = 0$, we get $f(-c) = f(c) + c - 1$, so $c \neq 0$.

It is easy to find the restriction f to A. Take $x = f(y)$ to obtain

$$f(x) = \frac{c+1}{2} - \frac{x^2}{2} \quad \text{for all} \quad x \in A. \tag{1}$$

The main step is to show that $A - A = \mathbb{R}$. Indeed, for $y = 0$, we get:

$$\{f(x-c) - f(x) \mid x \in \mathbb{R}\} = \{cx + f(c) - 1 \mid x \in \mathbb{R}\} = \mathbb{R}$$

because $c \neq 0$.

(The following is due to Lim Chong Jie.) With this we conclude that the given functional equation is equivalent to the following:

$$f(x - y) = f(y) + xy + f(x) - 1, \quad \text{for all } x, y \in \mathbb{R}.$$

By putting $y = 0$, we get $f(0) = 1$. Replacing y by x, we have

$$1 = 2f(x) + x^2 - 1, \quad \text{i.e., } f(x) = 1 - \frac{x^2}{2}.$$

It is easy to check that this function satisfies the given functional equation.