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The Arithmetic-Geometric Mean, Hypergeometric Series and π

The following article is the text of a talk given by the author at Tampines Junior College on 14 April 2000.

§ 1. The Arithmetic Geometric Mean

The *arithmetic mean* of two numbers a and b is defined as the "average" of the numbers, namely, $\frac{a+b}{2}$ while the *geometric mean* is given by $\sqrt{ab}$. The first relation we observe from these two means is that if a, b are non-negative, then

$$(*) \qquad \sqrt{ab} \leq \frac{a+b}{2}.$$

To see that (*) holds, simply observe that

$$0 \leq (a-b)^2 = a^2 + b^2 - 2ab = a^2 + b^2 + 2ab - 4ab = (a+b)^2 - 4ab.$$

This then implies that

$$ab \leq \left(\frac{a+b}{2}\right)^2,$$

which gives (*) since a and b are non-negative real numbers. Identity (*) known as the *Arithmetic-Geometric Inequality.*

We now fix two numbers $a_0 > 0$ and $b_0 > 0$, and define

$$a_n = \frac{a_{n-1} + b_{n-1}}{2} \quad \text{and} \quad b_n = \sqrt{a_{n-1} b_{n-1}}.$$

Notice that a_n is the arithmetic mean and b_n is the geometric mean of a_{n-1} and b_{n-1}, respectively.

Example 1

If we set $a_0 = b_0 = 1$, then the above iterations give

$$\begin{array}{lll} a_1 = (1+1)/2 = 1, & a_2 = 1, & a_3 = 1 \cdots \\ b_1 = \sqrt{1} = 1 & b_2 = 1, & b_3 = 1 \cdots \end{array}$$

So, $a_n = 1 = b_n$ for all n. In general if we start with the same a and b we always obtain the result $a_n = b_n = a = b$.

Example 2

Consider now the more interesting example. Set $a_0 = 1, b_0 = \sqrt{2}$. Then we have

$$\begin{array}{lll} a_1 = \frac{1}{2} + \frac{1}{2}\sqrt{2} & a_2 = \frac{1}{4} + \frac{1}{4}\sqrt{2} + \frac{1}{2}2^{1/4} & a_3 = \frac{1}{8} + \frac{1}{8}\sqrt{2} + \frac{1}{4}2^{1/4} + \frac{1}{2}\sqrt{(\frac{1}{2} + \frac{1}{2}\sqrt{2})\,2^{1/4}} \\ b_1 = 2^{1/4} & b_2 = \sqrt{(\frac{1}{2} + \frac{1}{2}\sqrt{2})\,2^{1/4}} & b_3 = \sqrt{(\frac{1}{4} + \frac{1}{4}\sqrt{2} + \frac{1}{2}2^{1/4})\sqrt{(\frac{1}{2} + \frac{1}{2}\sqrt{2})\,2^{1/4}}} \end{array}$$

One sees immediately that our numbers a_n and b_n get very complicated even for small n. At this point, everyone would have thought that the numbers a_n and b_n are uninteresting and "move on with their life" except probably the famous mathematician Carl F. Gauss. Gauss calculated several different sets of $\{a_n\}$ and

$\{b_n\}$ using different initial values a_0 and b_0 and noticed that as n gets large, a_n and b_n seems to "stabilize". So, for example, in Example 2, we have

n	a_n	b_n
1	$1.20710678118654752 44\cdots$	$1.18920711500272106 67\cdots$
2	$1.19815694809463429 56\cdots$	$1.19812352149312012 26\cdots$
3	$1.19814023479387720 91\cdots$	$1.19814023467730720 58\cdots$
4	$1.19814023473559220 74\cdots$	$1.19814023473559220 75\cdots$

So, Example 2 shows that a_n and b_n approach the **SAME LIMIT** $1.198\cdots$ as n approaches infinity. In the language of limits, we say that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 1.198140234735592207\cdots .$$

After observing this phenomenon, Gauss succeeded in showing using (*) that the limits of the sequence $\{a_n\}$ and $\{b_n\}$ exist and they coincide. In other words, there exist a number M such that a_n and b_n are both close to M when n is large. Since M depends on the initial values a_0 and b_0, Gauss defined the limit M as a function of two numbers, namely, $M := M(a,b)$ where $a_0 = a$ and $b_0 = b$. So our discussion above shows that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = M(a,b).$$

This number $M(a,b)$ is known as the *Arithmetic-Geometric Mean.*

§ 2. Two basic properties of the Arithmetic Geometric Mean

We now take a closer look at some of the elegant properties satisfied by the AGM $M(a,b)$. First, notice that

$$M(a,b) = M\left(\frac{a+b}{2}, \sqrt{ab}\right) = M(a_1, b_1).$$

This is because a_1 and b_1 may be treated as the initial value instead of a and b and the limit $\lim a_n = \lim b_n$ are not affected. So, we have the relation

$$M(a,b) = M\left(\frac{a+b}{2}, \sqrt{ab}\right). \tag{2.1}$$

Next, notice that if we start with $a^* = ca$ and $b^* = cb$, with $c > 0$, our sequence would be

$$a_1^* = c\frac{a+b}{2} = ca_1, \quad \text{and} \quad b_1^* = c\sqrt{ab} = cb_1$$

and in general, we have

$$a_n^* = ca_n \quad \text{and} \quad b_n^* = cb_n.$$

In other words, a_n^* tends to the product of c and the limit of a_n, namely,

$$\lim_{n\to\infty} a_n^* = cM(a,b).$$

But by Gauss' definition of M,

$$\lim a_n^* = M(ca, cb).$$

Hence, we have the second relation, namely, if $c > 0$, then

$$M(ca, cb) = cM(a, b). \tag{2.2}$$

Relation (2.2) shows that

$$M(a, b) = aM(1, b/a). \tag{2.3}$$

Applying (2.3) to (2.1), we immediately deduce that

$$aM(1, b/a) = \frac{a+b}{2} M\left(1, \frac{2\sqrt{b/a}}{1+b/a}\right),$$

which implies that

$$M(1, x) = \frac{1+x}{2} M\left(1, \frac{2\sqrt{x}}{1+x}\right), \tag{2.4}$$

where $x = b/a$. This is clearly an interesting relation if we treat $M(1, x)$ as a function of x. Gauss realized the beauty of this identity and together with his observation that

$$\frac{1}{M(1, \sqrt{2})} \quad \text{and} \quad \frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-t^4}} dt$$

are equal up to a total of 11 decimal places, he was able to show, using (2.4), that

$$\frac{1}{M(1, 1-x^2)} = \frac{2}{\pi}\int_0^{\pi/2} \frac{1}{\sqrt{1 - x^2\sin^2 t}}\, dt. \tag{2.5}$$

(Note that $\left(\frac{2}{\pi}\int_0^1 \frac{1}{\sqrt{1-t^4}} dt\right)^{-1} = 1.198140234735592207\cdots$.)

At this point, let me quote Gauss' comments extracted from his diary:

> *this result ((2.5)) will surely open up a whole new field of analysis.*

Gauss was right and this new field of analysis is now known as the *theory of elliptic functions.*

§ 3. A sketch of the proof of (2.5)

Gauss' original proof of (2.5) was rather complicated. He established (2.5) by assuming that $1/M(1, \sqrt{1-x^2})$ is an analytic function in x and hence, has a power series expansion. He then established the coefficients of the power series using (2.4). I would like to present here a simple proof of (2.5). The most non-trivial technique is probably integration by substitution.

Define

$$T(a,b) := \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}.$$

Then, substituting $t := b\tan\theta$ yields

$$T(a,b) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}}.$$

On substituting $u := \frac{1}{2}(t - ab/t)$, we find that

$$T(a,b) = T\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

This means that

$$T(a,b) = T(a_1,b_1) = T(a_2,b_2) = \cdots = T(a_n,b_n),$$

where a_n and b_n are defined as in Section 1. Since a_n and b_n tend to $M(a,b)$ as n gets large, we may conclude that

$$T(a,b) = T(M(a,b), M(a,b)).$$

But

$$T(M(a,b),M(a,b)) = \frac{1}{M^2(a,b)}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{dt}{1+(t/M(a,b))^2} = \frac{1}{M(a,b)}.$$

Therefore,

$$\frac{1}{M(a,b)} = T(a,b).$$

Substituting $a = 1$, and $b = \sqrt{1-x^2}$ yields

$$\frac{1}{M(1,\sqrt{1-x^2})} = \frac{2}{\pi}\int_0^{\pi/2}\frac{1}{\sqrt{1-x^2\sin^2 t}}\,dt,$$

which is (2.5).

§ 4. Binomial Theorem and Hypergeometric Series

The binomial theorem states that if n is an integer then

$$(1-u)^n = \sum_{k=0}^{n}\binom{n}{k}(-1)^k u^k,$$

where

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{(n-k+1)(n-k+2)\cdots n}{1\cdot 2\cdots k}.$$

We now introduce the notation

$$(a)_m := (a)(a+1)\cdots(a+m-1). \tag{4.1}$$

Then with this new notation,

$$\binom{n}{k} = (-1)^k \frac{(-n)_k}{(1)_k}.$$

Note that although the symbol $\binom{n}{k}$ makes sense only when n is an integer, our notation (4.1) makes sense even if a is not an integer. Thus, we may rewrite our binomial theorem as

$$(1-u)^a = \sum_{k=0}^{\infty} \frac{(-n)_k}{(1)_k} u^k.$$

Note that when a is an integer, the sum is finite and so, we obtain our original binomial theorem. The above allows us to have a series expansion for $(1-u)^a$ even if a is not an integer.

Example 3

$$(1-u)^{-1/2} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(1)_k} u^k.$$

Now, applying the above with $u = x^2 \sin^2\theta$, we find that

$$\frac{1}{\sqrt{1-x^2\sin^2\theta}} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(1)_k} x^{2k} \sin^{2k}\theta.$$

Therefore,

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k}{(1)_k} \left(\int_0^{\pi/2} \sin^{2k}\theta d\theta\right) x^{2k}.$$

Upon using the well-known integrals

$$\int_0^{\pi/2} \sin^{2k}\theta d\theta = \frac{\pi}{2}\frac{\left(\frac{1}{2}\right)_k}{(1)_k},$$

we conclude that

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{2}\right)_k^2}{(1)_k^2} x^{2k}. \tag{4.2}$$

The function on the right is denoted as ${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x^2)$. This is an example of the Gaussian hypergeometric series. The general definition of the Gaussian hypergeometric series is given by

$${}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}.$$

At first sight, this may look strange. But one could easily verify that

$$\begin{aligned}(1+z)^a &= {}_2F_1(-a,b;b;-z),\\ \arcsin z &= z\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};z^2\right),\\ \arctan z &= z\,{}_2F_1\left(\frac{1}{2},1;\frac{3}{2};-z^2\right),\end{aligned}$$

and

$$\ln(1+z) = z\,{}_2F_1\left(1,1;2;-z\right).$$

The hypergeometric series is a very important class of special functions and from the above, it is certainly a generalization of the functions we know. Note that we are led to this function by a simple consideration of the arithmetic and geometric mean!

I end this Section with an interesting identity, namely,

$${}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\frac{1}{2}\right) = \left(\sum_{k=0}^{\infty} e^{-\pi k^2}\right)^2.$$

Note also that by the idenfication (4.2), we are actually evaluating $1/(\sqrt{2}M(1,\sqrt{2}))$ on the left hand side.

§ 5. A transformation formula for ${}_2F_1(\frac{1}{2},\frac{1}{2};1;z)$

We have found in Section 4 that

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;x^2\right).$$

On the other hand, we found in Section 3 that

$$\frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x^2\sin^2\theta}} = \frac{1}{M(1,\sqrt{1-x^2})},$$

or upon letting $x=\sqrt{1-t^2}$,

$$\frac{1}{M(1,t)} = {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-t^2\right).$$

Now, M satisfies the transformation formula (2.4), namely,

$$M(1,t) = \frac{1+t}{2}M\left(1,\frac{2\sqrt{t}}{1+t}\right).$$

Hence, ${}_2F_1$ satisfies

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-t^2\right) = \frac{2}{1+t}\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;\left(\frac{1-t}{1+t}\right)^2\right), $$

which is indeed an elegant transformation formula. If we set $t = (1-s)/(1+s)$, then we obtain

$$ {}_2F_1\left(\frac{1}{2},\frac{1}{2};1;1-\left(\frac{1-s}{1+s}\right)^2\right) = (1+s)\,{}_2F_1\left(\frac{1}{2},\frac{1}{2};1;s^2\right). $$

Now the expression $1-(1-s)^2/(1+s)^2$ is a very interesting expression. I will end my talk with the following new result, which is related to our topic today :

Let $k_0 = 0$ and $s_0 = \dfrac{1}{\sqrt{2}}$. Set

$$ s_n = \sqrt{1-\left(\frac{1-s_{n-1}}{1+s_{n-1}}\right)^2} $$

and

$$ k_n = \left(\frac{2}{s_{n-1}+1}\right)^2 \left\{2^{n-1}(1-s_{n-1})s_{n-1} + k_{n-1}\right\}, $$

then k_n^{-1} tends to π quadratically.

The following shows the convergence of k_n:

n	$k_n^{-1} - \pi$
1	$0.3761\cdots$
2	$0.1189\cdots \times 10^{-2}$
3	$0.7987\cdots \times 10^{-8}$
4	$0.1905\cdots \times 10^{-18}$
5	$0.5582\cdots \times 10^{-40}$
6	$0.2430\cdots \times 10^{-83}$

Acknowledgement : This article is based on a talk delivered to a group of students at Tampines Junior College. The author wishes to thank Mdm Tan Lay Hoon for the invitation.

Remarks : Most of the materials in this article are from [1]. It is the author's hope that the readers will be motivated to read [1] (which happens to contain many charming identities) after reading this article.

References

1. J.M. Borwein and P.B. Borwein, *Pi and the AGM*, John Wiley, New York, 1987.