Contest

1. Prizes in the form of book vouchers will be awarded to the first received best solution(s) submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

3. Solutions should be typed and sent to : The Editor, Mathematics Medley, c/o Department of Mathematics, National University of Singapore, 2 Science Drive 2, Singapore 117543; and should arrive before 31 October 2000.

4. The Editor's decision will be final and no correspondence will be entertained.

Problems Corner

Problem 1.

Find all pairs of positive integers (m, n) such that $m^2 + 2000 = 6^n$.

(Prize One $100 book Voucher)

Problem 2.

ABCD is a quadrilateral inscribed in a circle with centre at O, and P is the intersection of AC and BD. Let O_1, O_2, O_3 and O_4 be the circumcentres of the triangles PAB, PBC, PCD and PDA respectively. Prove that the lines OP, O_1O_3 and O_2O_4 are concurrent.

(Prize One $100 book Voucher)

Solutions to the problems in volume 26 No. 2 December 1999.

Problem 1

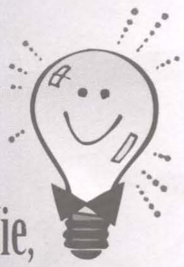
Let x_1, x_2, x_3, x_4 denote the four roots of the equation

$$x^4 - 18x^3 + kx^2 + 90x - 2000 = 0$$

where k is a constant. If $x_1x_2 = 50$, find the value of k.

PRIZE *one $100 book voucher*

Solution by
Lim Chong Jie,
Temasek Junior College,
Class 05/98.



The answer is $k = 87$.

From the given equation, we obtain the following equations:

$$x_1 + x_2 + x_3 + x_4 = 18 \qquad (1)$$
$$x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = k \qquad (2)$$
$$x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = -90 \qquad (3)$$
$$x_1x_2x_3x_4 = -2000 \qquad (4)$$

Given $x_1x_2 = 50$, by (4), $x_3x_4 = -40$.

Therefore (3) becomes $50(x_3 + x_4) - 40(x_1 + x_2) = -90$.

From (1) & (3), we obtain $x_1 + x_2 = 11$ and $x_3 + x_4 = 7$.

From (2), $x_1x_2 + (x_1 + x_2)(x_3 + x_4) + x_3x_4 = k$.

Substituting the known values, we obtain $k = 87$.

Solved also by Tan Tze Jwee Glynn, Anglo-Chinese Junior College, Class 2SA1; S. Thiagarajah; Lim Yin, Victoria Junior College, Class 99S22; Julius Poh Wei Quan, AngloChinese (Independent) School, Class 4.14; and Zhang Nan Ruo, Kranji Secondary School, Class 4D.

Three incorrect solutions were received.

Editor's note:

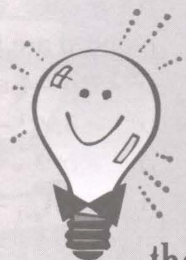
The prize was shared equally between Lim Chong Jie and Tan Tze Jwee Glynn.

For each positive integer n, let A_n be the (unique) positive integer which satisfies

$$\left(\sqrt{3}+1\right)^{2n} \le A_n < \left(\sqrt{3}+1\right)^{2n} + 1.$$

Prove that A_n is divisible by 2^{n+1}.

PRIZE *one $100 book voucher*



Solution by the Editors

For each positive integer n, we define

$$B_n = (\sqrt{3}+1)^{2n} + (\sqrt{3}-1)^{2n}.$$

Lemma 1. For each positive integer n, B_n is a positive integer.

Proof.

$$\begin{aligned} B_n &= (\sqrt{3}+1)^{2n} + (\sqrt{3}-1)^{2n} \\ &= \sum_{k=0}^{2n} \binom{2n}{k} (\sqrt{3})^k + \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} (\sqrt{3})^k \\ &= 2\sum_{k=0}^{n} \binom{2n}{2k} 3^k \end{aligned}$$

$\therefore$ B_n is a positive integer.

Lemma 2. For each positive integer n, we have

$$A_n = B_n.$$

Proof.

$$\begin{aligned} &\because \quad 0 < \sqrt{3}-1 = \frac{2}{\sqrt{3}+1} < 1 \\ &\therefore \quad 0 < (\sqrt{3}-1)^{2n} < 1 \\ &\therefore \quad (\sqrt{3}+1)^{2n} < B_n < (\sqrt{3}+1)^{2n} + 1 \\ &\because \quad -(\sqrt{3}+1)^{2n} - 1 < -A_n \le -(\sqrt{3}+1)^{2n} \\ &\therefore \quad -1 < B_n - A_n < 1 \end{aligned}$$

$\because$ B_n is an integer by Lemma 1 and A_n is given to be an integer

$\therefore$ $B_n - A_n$ is also an integer and hence

$$\begin{aligned} &B_n - A_n = 0 \\ \therefore \quad &A_n = B_n. \end{aligned}$$

Solutions to the problems in volume 26 No. 2 December 1999.

Lemma 3. For each positive integer n, we have

$$A_{n+2} = 8A_{n+1} - 4A_n.$$

Proof. By Lemma 2, we have

$$\begin{aligned} A_n &= B_n \\ &= (\sqrt{3}+1)^{2n} + (\sqrt{3}-1)^{2n} \\ &= (4+2\sqrt{3})^n + (4-2\sqrt{3})^n. \end{aligned}$$

Let $p = 4+2\sqrt{3}$, $q = 4-2\sqrt{3}$.

$\therefore$ $p + q = 8$ and $pq = 4$.

$\therefore$ p and q are the roots of the quadratic equation $x^2 - 8x + 4 = 0$.

$\therefore$ $p^2 = 8p - 4$ and $q^2 = 8q - 4$.

$\therefore$ $p^{n+2} = 8p^{n+1} - 4p^n$ and $q^{n+2} = 8q^{n+1} - 4q^n$.

$\therefore$ $p^{n+2} + q^{n+2} = 8(p^{n+1} + q^{n+1}) - 4(p^n + q^n)$.

i.e. $A_{n+2} = 8A_{n+1} - 4A_n$.

Solution by the Editors

We shall now prove by induction that A_n is divisible by 2^{n+1}.

We have $A_1 = 8$ and $A_2 = 56$ and they are divisible by $2^{1+1} = 4$ and $2^{2+1} = 8$ respectively. Now assume that for each $k \le n$, we have 2^{k+1} divides A_k.

$\therefore$ $A_n = 2^{n+1}x$ and $A_{n-1} = 2^n y$ for some positive integers x and y.

$\therefore$ By Lemma 3, we have

$$\begin{aligned} A_{n+1} &= 8A_n - 4A_{n-1} \\ &= 8(2^{n+1}x) - 4(2^n y) \\ &= 2^{n+2}(4x - y). \end{aligned}$$

$\therefore$ A_{n+1} is divisible by 2^{n+2}.

$\therefore$ *By induction,* A_n is divisible by 2^{n+1}.

One incorrect solution was received.