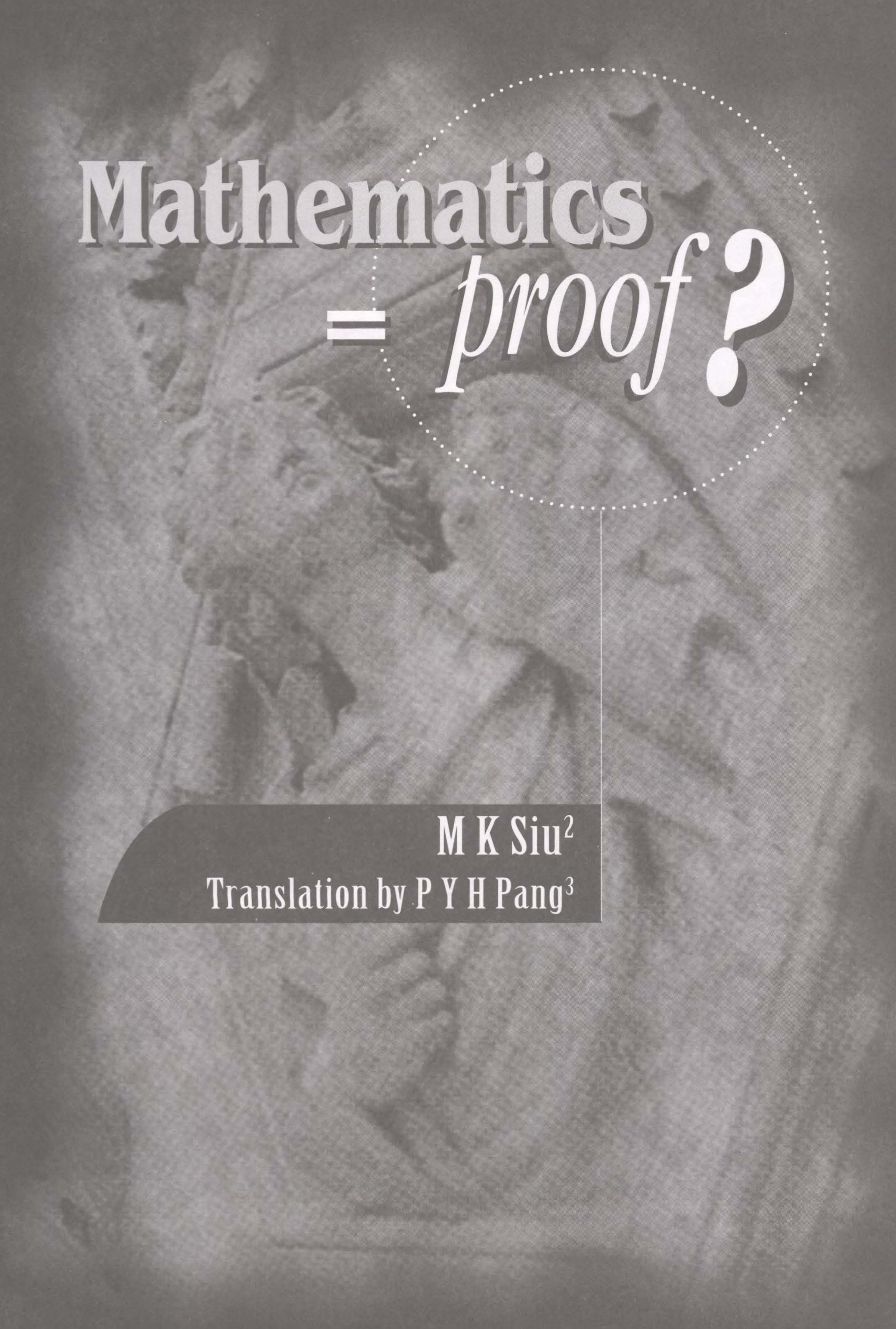

Mathematics = *proof* ?

M K Siu[2]

Translation by P Y H Pang[3]

What is a Mathematical Proof?

I suppose most readers know what a mathematical proof is. Let's say we wish to know if the mathematical statement "If *p* then *q*" holds. The process of determining the truth or falsehood of this statement using only (i) *fundamental concepts* (definitions), (ii) *fundamental hypotheses* (axioms), (iii) *previously established results* (theorems), and (iv) *logically correct inference* is called a *mathematical proof.*

Proposition 47 in Book I of Euclid's *Elements* reads: "In right-angled triangles the square on the side subtending the right angle is equal to the sum of the squares on the sides containing the right angle." [4]This ancient and important result, too well known to the readers to warrant a proof here, did intrigue a famous 17th century English philosopher. At the age of 40 and never having studied geometry before, Hobbes was said to have come across this theorem quite by chance in his friend's study. His curiosity urged him to read on to the proof. The proof, however, quoted a previous theorem whose proof in turn quoted a previous theorem and so on. After several hours' work, he was finally convinced of the truth of Proposition 47, and thus started his life-long love for geometry.

Alas, those like Hobbes who love mathematics for its logical reasoning are a rare breed; rather, most shy away from the subject because they perceive it as all logical deductions and tedious calculations. Nevertheless, whether you love or hate mathematics, you would probably agree with Hobbes that a mathematical proof starts with certain basic assumptions or axioms and arrives at the conclusion through a series of logically correct deductions. Some people even equate mathematics with proofs. But is there more to mathematics than proofs?

Thus Spake the Philosophers

In many ways, the mathematical proof was thrust onto centrestage as a result of a crisis in the foundations of mathematics in the early part of the 20th century, in particular under the strong influence of logicism and formalism. To cut a long story short, let me just produce a few relevant quotes:

Mathematics in its widest significance is the development of all types of formal, necessary, deductive reasoning. - A. N. Whitehead (1898)

Pure mathematics is the class of all propositions of the form 'p implies q' where p and q are propositions. - B. Russell (1903)

Mathematics is the motley of techniques of proof. - L.Wittgenstein (1956)

Mathematics is the science of making necessary conclusions. - B. Peirce (1881)

[1] Editor's Note: This is a translation of an article (in Chinese) by Prof M K Siu published in *Shuxue Chuanbo (Mathmedia)* **16, 4** (1992), 50-58. The Singapore Mathematical Society wishes to thank Prof Siu for allowing this translation to be published in the *Mathematical Medley.*

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[3] P Y H Pang is Associate Professor in Mathematics at the National University of Singapore.

[4] Translator's Note: Generally known in the West as Pythagoras'Theorem, this result, however, has been known in China before the time of Pythagoras as the *Gou Gu* Theorem. The latter name is used in the original Chinese article.

Perhaps it would be fairer to point out that the centre of the philosophical debate was really the consistency of mathematics as an academic discipline, and not a battle of personal views on the nature of mathematics as an intellectual activity. Nevertheless, this philosophical debate seemed to have exerted influence that far exceeded its original intention and, by focusing on the mathematical proof, scrutinizing its nature and dissecting its structure, shaped the general opinion that the main job of a mathematician is to prove theorems, and proving theorems is an exercise in logic.

To say that a mathematician's job is to prove theorems is akin to saying that a writer's job is to construct sentences, a composer's job is to assemble notes and an artist's job is to draw and colour. Hence, Li Bai's poems would just be a compilation of phrases, Beethoven's symphonies a mere ensemble of notes, and Qi Baishi's paintings an arrangement of lines! If literature, music and art are capable of expressing ideas and emotions, why can't mathematics have its own sense of aesthetics? Acknowledging that perception of aesthetics is an individual experience, I have no intention of pursuing a discourse on this aspect. What I would like to discuss, rather, is whether, as an intellectual activity, "Mathematics = Proof" is a fair statement.

Did Euclid Discover Pythagoras' Theorem?

Let us go back to Euclid's Proposition 47 (Pythagoras' Theorem). The proof presented in the *Elements* seems to be the first recorded in history. But does that mean that people did not know Pythagoras' Theorem before then (4th century BC)?

Figure 1

$(d/h)^2$	h	b	d
1.9834027	120	119	169
1.9491585	3456	3367	11521 (4825)
1.9188021	4800	4601	6649
1.8862478	13500	12709	18541
1.8150076	72	65	97
1.7851928	360	319	481
1.7199836	2700	2291	3541
1.6927093	960	799	1249
1.6426694	600	541 (481)	769
1.5861225	6480	4961	8161
1.5625000	60	45	75
1.4894168	2400	1679	2929
1.4500173	240	25921 (161)	289
1.4302388	2700	1771	3229
1.3871604	90	56	53 (106)

In the Columbia University Museum lies a clay tablet named Plimton 322 which dates from the Babylonian era of 19 centuries BC, 1500 years before Euclid. In the 1943 catalogue of the Museum, this clay tablet, on which a few lines of numbers were inscribed (see Figure 1 above), was classified as "commercial account". Two years later, two prominent historians of mathematics Neugebauer and Sachs made the following startling discovery: The content of Plimton 322 is a list of Pythagorean triplets, i.e., triplets of positive integers $\{h, b, d\}$ such that $h^2 + b^2 = d^2$. Actually, the list only contains the values of b and d, without h. However, the values of $(d/h)^2$ are given in the leftmost column. What is more, adjacent numbers in this leftmost column differ by about 0.03. Look at the fourth line for the Pythagorean triplet {13500, 12709, 18541}. Do you really think the ancient Babylonians were ignorant of Pythagoras' Theorem and just stumbled upon these triplets?

Refer also to the ancient Chinese text *Zhoubi Suanjing* (c. 1st century BC). There is a passage that gives the following proof of Pythagoras' Theorem: Rotate the given right-angled triangle (ABC) about the centre of the square on the hypotenuse to form triangles *FCY*, *GYX* and *EXB* as in the diagram (Figure 2).

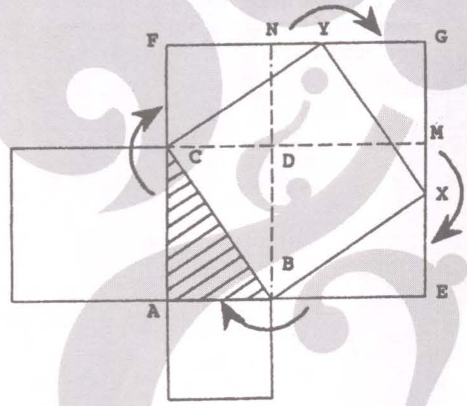


Figure 2

Then, it is easy to see that

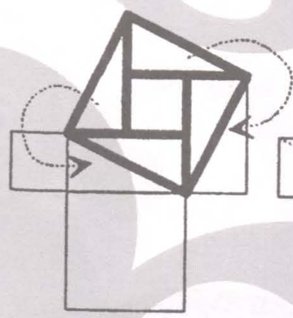
$$\square AFGE = \square CDNF + \square BEMD + 2 \square ABDC,$$

and

$$\square AFGE = \square BXYC + 4 \Delta ABC.$$

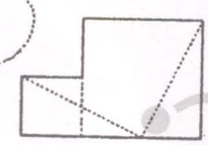
From this, one sees that the area of the square on the hypotenuse (*BC*) is the sum of the areas of the squares on the other two sides (*AB* and *CA*) of the right-angled triangle (*ABC*).[5]

During the time of the Three Kingdoms (c. 3rd century AD) in China, the Wu mathematician Zhao Shuang provided a similar proof in his annotation of *Zhoubi Suanjing* (Figure 3a). Another similar idea was proposed by the 12th century Indian mathematician Bhaskara (Figure 3b). It is amusing to note that, besides the diagram, Bhaskara's proof consists only of a single exclamation: "Behold!" These proofs are all different from the one in the *Elements*.



a

Figure 3a



b

Figure 3b

Thus, we can see that the content of Pythagoras' Theorem neither started nor ended with Euclid's proof. On the contrary, it is after a statement has been thoroughly understood that a rigorous proof can be found. This has manifested repeatedly in the history of mathematics, the development of calculus being a typical example.[6] The 19th century English mathematician de Morgan said thus: "The moving power of mathematical invention is not reasoning but imagination."

[6] Editor's Note: The reader may refer to Prof Siu's articles "The Story of Calculus (I and II)", *Mathematical Medley* Volume 23 No. 1 and 2 (1996).}

When Do You Believe a Mathematical Statement?

No doubt, some mathematical statements are "self-evident". For example: perpendiculars dropped from two vertices of a triangle meet at a point; the sum of squares of two real numbers is nonnegative; opposite angles (formed by two intersecting straight lines) are equal; two lines that are each parallel to a third straight line are themselves parallel to each other. And then, there are statements whose validity is not quite as obvious but are nonetheless convincing upon sufficient observation and experimentation. Examples of this class: perpendiculars dropped from the three vertices of a triangle are concurrent; the sum of squares of two real numbers is not less than twice the product of the two numbers. Also, there are statements that appear to be abstract, but are believable by virtue of a physical interpretation. An example is: if the derivative of a function is everywhere zero, then the function is a constant. The physical interpretation of this statement is that a particle with zero velocity stays put! Alas, there are statements that do not belong to any of the above three classes. How are we to be convinced of them if not for proofs?

When we trace a sophisticated theorem to its origin, we often find its formulation to have been prompted by certain "circumstantial evidences" which render such a result plausible. Let me illustrate this with an example. Prime numbers seem to appear rather haphazardly. You may wonder: Between any two numbers a and b, which numbers are prime? How many of them are there? How far apart do they appear? Let us consider the following observations: Between 0 and 99 (100 numbers), there are 25 prime numbers of which eight pairs differ by only 2 in value; between 9,999,900 and 10,000,000 (also 100 numbers), there are 9 prime numbers of which two pairs differ by only 2 in value; but between 10,000,000 and 10,000,100 (again 100 numbers), there are only 2 primes and their difference is 60. What appears to be totally chaotic turns out to possess some order after all, and this was observed by some 18th and 19th century mathematicians such as Legendre and Gauss. Let us observe the following table in which $\pi(N)$ denotes the number of prime numbers between 1 and N:

Figure 4

N	$\pi(N)$	$\pi(N)/N$
10	4	0.4
10^2	25	0.25
10^3	168	0.168
10^4	1229	0.1229
10^5	9592	0.09592
10^6	78498	0.078498
10^7	664577	0.0664577
10^8	5761455	0.05761455
10^9	50847354	0.050847354
10^{10}	455052512	0.0455052512

Note that the right-most column gives the density of prime numbers. Multiply these numbers by 1,2,3,... respectively, i.e., log N, we will get a list of numbers converging to a number c between 0.4 and 0.5. Thus,

$$\frac{\pi(N)}{N} \times \log N \sim c, \text{ or } \pi(N) \sim c \times \frac{N}{\log N}.$$

In fact, $c = \log e$ where $e = 2.71828...$ is nothing else but the base of the natural logarithm. Thus,

$$\pi(N) \sim \frac{N}{\log_e N}.$$

This relationship[7] was finally proved at the end of the 19th century and was known as the *Prime Number Theorem*. Thus, the global distribution of prime numbers follows a simple and elegant rule even though their local distribution is poorly understood.

[7] Translator's Note: The symbol "~" means that the left hand side and the right hand side get closer and closer to each other as N gets larger and larger.

Let us push this one step further and suppose that the local distribution of primes is random subject to the Prime Number Theorem. That is to say, let us hypothesize that whether a number is prime or not is determined by the toss of a coin that is loaded in such a way that, when tossed N times, the probability of head is $1/\log_e N$. Then, when head appears at the kth toss among N tosses, k will be a prime number. (I beg the readers' indulgence in this ridiculous model and urge them to read on.)

Now, removing the number 2, all other prime numbers are odd and differ from the closest prime by at least 2. A pair of primes that differ by 2 is called twin primes, examples are 3 and 5, 5 and 7, 11 and 13, etc. There is a famous conjecture regarding twin primes, which is that there are infinitely many of them (a consequence of this would be that large prime numbers need not be far apart). Let us investigate this using the coin-toss model outlined above. To be more specific, we ask the question: What is the probability that two numbers k and $k + 2$ between 1 and N be primes, i.e., that heads appear at the kth and $(k + 2)$th tosses of the coin? A rough calculation shows that that probability is $(1/\log_e N)^2$, and hence, we expect to have $N/(\log_e N)^2$ twin primes between 1 and N. Without going into details, we simply mention that actually tossing head at the kth and $(k + 2)$th times are not really independent events, and thus a more accurate answer should be

$$\frac{(1.32...)\times N}{(\log_e N)^2} \qquad (*)$$

twin primes between 1 and N. The following table shows that this is in fact quite close to the actual answer.

Figure 5

Range	No. of twin primes by coin-toss model	Actual number
$10^9 - 10^9 + 150{,}000$	466	461
$10^{10} - 10^{10} + 150{,}000$	389	374
$10^{11} - 10^{11} + 150{,}000$	276	309
$10^{12} - 10^{12} + 150{,}000$	276	259
$10^{13} - 10^{13} + 150{,}000$	208	221
$10^{14} - 10^{14} + 150{,}000$	186	191
$10^{15} - 10^{15} + 150{,}000$	161	166

Considering how ridiculous the coin-toss model is, this coincidence is shocking! Could the formula (*) really give the correct (asymptotic) distribution of twin primes? (If so, the Twin Prime Conjecture would be settled.) The "circumstantial evidence" given above is certainly in its favour. However, no mathematician would accept that it is a proof.

Why Do We Still Need Proofs?

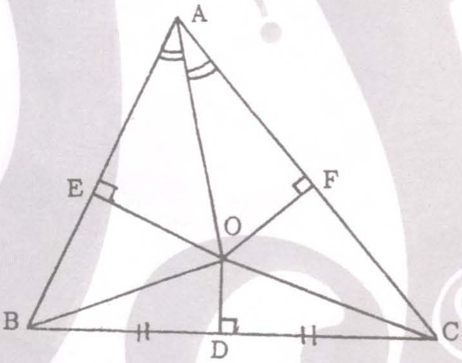


Figure 6a

Not only the layperson, even other scientists fail to appreciate why mathematicians take mathematical proofs so seriously. When his assistant the young mathematician Harish-Chandra told Dirac that he was troubled as he could not find the proof even though he was sure his answer was correct, the eminent English physicist said: "I don't care about proofs, I want to know the truth!"

Most "circumstantial evidences" belong to one of the following categories: geometric observation, inductive evidence, and analogy. Let us look at these one by one.

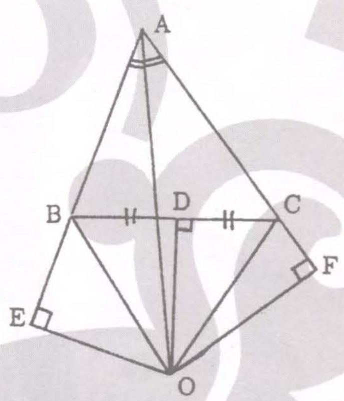


Figure 6b

First, geometric observation. In 1908, Klein put forward the following widely quoted example in which he "proved" that every triangle was isosceles! It goes as follows: For a triangle *ABC*, let *D* be the midpoint of the side *BC* and let the bisector of $\angle BAC$ intersect the perpendicular bisector of the side *BC* at *O*. From *O*, drop perpendiculars to the sides *AB* and *AC*, meeting them at *E* and *F* respectively (Figure 6a). Then, from the congruence of the pairs of triangles *AOE*, *AOF* and *BOD*, *COD*, one can easily deduce that the pair *BOE*, *COF* are also congruent. It follows that $AE = AF$, $BE = CF$, and hence *ABC* is isosceles. Some of you may argue that the point *O* may lie outside *ABC*, but a similar argument seems to go through even in that case (see Figure 6b). So what went wrong? I am sure the careful reader can spot it by drawing an accurate diagram. We note, however, the obvious shortcomings of relying on accurate diagrams. As an axiomatic subject in which theorems are obtained through deductive reasoning, Euclidean geometry calls for accurate logical arguments supplemented by rough diagrams (to aid our intuition), rather than rough explanations (based on intuition) supplementing accurate diagrams. This example shows that the precision and logic associated with Euclidean geometry have their rightful place in establishing mathematical statements.

Next, let us look at inductive evidence. Consider the following question: for $y \neq 0$, can $1+1141y^2$ be a perfect square? This is the same as asking whether the equation $x^2 - 1141y^2 = 1$ has an integer solution. This equation was studied by Fermat in the 17th century, but somehow Euler mistakenly attributed it to the English mathematician Pell. The name Pell's equation has since stuck with equations of this type. Perhaps you are patient enough to try every integer from $y = 1$ to $y = 10^7$, but still you will not find a solution. However, mathematicians have proved that not only do solutions exist, there are in fact an infinite number of them. The smallest y appearing in a solution happens to be 30,693,385,322,765,657,197,397,208 which is about 3×10^{25}! The corresponding x is 1,036,782, 394,157,223,963,237,125,215 which is about 10^{27}.

Finally, let us look at analogy. In the 3rd century BC, Archimedes proved the area formula for an ellipse: $A = \pi ab$, where a and b are the semi-major and semi-minor axis respectively. If $a = b = r$, this reduces to the well-known formula for the area of a circle, namely, $A = \pi r^2$. Now, consider the square whose sides are tangential to the circle. The ratio of the area of the circle to that of its tangential square is $\pi : 4$, and this ratio happens to coincide with that of the perimeters of the circle and the square. So, by analogy, it seems perfectly reasonable to guess that, as the ratio of the area of an ellipse to that of its tangential rectangle is $\pi : 4$, it should also be equal to the ratio between their perimeters. In this case, since the perimeter of the rectangle is $4(a + b)$, the perimeter of the ellipse would be

$$\frac{\pi}{4} \times 4(a+b) = \pi(a+b).$$

(Note that when $a = b = r$, this reduces to the correct formula for the perimeter of a circle.) Indeed the 13th century Italian mathematician Fibonacci did propose this formula, which of course we now know is wrong. As a matter of fact, the perimeter of an ellipse is neither simple to compute nor expressible in closed form and has eluded the grasp of mathematicians until the late 19th century.

But How Reliable Are Proofs?

University of California at Berkeley professor Berlekamp's book *Algebraic Coding Theory* is a classic reference in the field and has been translated into many languages. In the preface, he promised to pay US$1 to anyone who pointed out a mistake, large or small, for the first time. I first read this book in the winter of 1978 and discovered that one of the proofs in chapter 4 contained an error, which I rectified, and notified the author. He wrote back half a month later, and said, as I had expected, that the dollar had already been claimed 9 years ago. In the letter he also appended a list of corrigenda that ran 13 pages long and contained some 250 items. He also said that he was still paying 3–4 dollars every year after all these years. Yet, this in no way diminishes the merit of the book!

The following sensational news was reported in a 1945 issue of *Time* magazine: The American mathematician Rademacher had announced a solution to one of the most famous of all mathematical problems - the Riemann Hypothesis. In the spring of 1986, the *New York Times* reported with quite a bit of fanfare that the English mathematician Rourke and his Portugese colleague Rego had solved yet another famous problem - the Poincaré Conjecture. Again, the March 1988 issue of *Time* magazine reported that the Japanese mathematician Miyaoka had achieved the ultimate - proving Fermat's Last Theorem. All these proofs, however, were later discovered to contain irreparable flaws; even today, all these problems remain unsolved (Translator's Note: Fermat's Last Theorem has since been proved by Wiles and Taylor in 1995). Yet, no one slights these mathematicians for their mistakes, which may in fact contribute positively to the eventual solutions of these problems.

Such examples abound in the history of mathematics. Take Fermat's Last Theorem. The erroneous report the French mathematician Lamé made to the Paris Academy of Science on 1 March 1847 had an important influence on the development of number theory. Take also the Four Colour Problem[8] posed in 1878 by the English mathematician Cayley (the problem in fact originated with a young man named Guthrie in 1852 and was brought to the public's attention by the English mathematician de Morgan). In 1879, the Englishman Kempe, a lawyer by training, proposed a solution, only to be invalidated by his compatriot Heawood 11 years later. Kempe's (erroneous) solution, however, provided the basis for subsequent research on this problem. In fact, its final positive resolution in 1976 was based on Kempe's approach. This final solution, which included 1200 hours of machine computation, raised another controversy: Can a computer proof be accepted as a mathematical proof?

[8] Translator's Note: The problem asks whether every map drawn on the plane can be coloured using only 4 colours.

I have heard on the grapevine (unconfirmed, of course), that, according to one editor of the *Mathematical Reviews*, almost half of all published proofs are wrong, even though the theorems are correct!

Who Checks the Proofs?

In principle, there exists a system by which all mathematical concepts and theories can be put into formal or symbolic representation. For example, 1 + 1 = 2 has the representation

$$= (+(s(0),\ s(0),\ s(s(0))))\text{.}$$

In this system, all proofs appear as a finite sequence of such formal statements. Then, the validation of a proof reduces to checking whether this sequence follows the syntax of the formal system, and can be accomplished mechanically, quite devoid of human involvement. This was indeed the grand scheme proposed by Hilbert in the 1920's and 30's, in the hope that this would settle once and for all the question of the *consistency* of mathematics.

Does this grand scheme of formalism really work? A student of the Polish mathematician Steinhaus was supposed to have written down a proof of the Pythagoras theorem using the system found in Hilbert's *Foundations of Geometry*. The proof filled 80 pages! Tedium notwithstanding, the fatal blow came in 1931 when the Austrian mathematician Gödel published the following two earth-shattering theorems:

1. Any formal system that is compatible with arithmetic is incomplete, i.e., there are statements within the system that cannot be proved or disproved by the system.

2. Any formal system that is compatible with arithmetic cannot establish its own consistency.

How Do Mathematicians Work, Really?

Do most mathematicians work within a formal system? Not really. Actually, most of the time they only provide the main points of their arguments in a proof. Of course, their writing contains a multitude of symbols and formulae, but they are just shorthand notations, and basically have nothing to do with the kind of formal system that we have been discussing. In fact, "devoid of human involvement" is just about the furthest from the truth about their work. Proofs are written by humans, studied by humans, and judged by humans.

There are, however, some proofs whose length and complexity challenge the most patient and meticulous. The classification of finite groups is a good example. The problem, which originated around 1890, asks how many distinct groups of order N there are. It took the collective efforts of numerous mathematicians, producing well over 5000 pages of work over a century, to solve. It is doubtful whether anyone has really scrutinized these 5000 over pages in entirety!

In the last couple of decades, "computer proofs" have begun to appear. The earliest famous example was the proof of the Four Colour Problem in 1976 by the American mathematicians Haken and Appel we mentioned earlier. Assisted by Koch, they used 1200 hours of computer time to complete the proof. More recently, in the winter of 1988, a team at Concordia University in Canada, led by Clement Lam, proved the non-existence of finite projective planes of order 10. Using a CRAY-1A supercomputer from the US Institute of Defense Analyses (IDA) as well as VAX machines at Concordia, they spent three years to chalk up a total of over 2000 hours of computer time to complete the proof. No one could guarantee that no mistake had been made, and, if a mistake was indeed made, it would be difficult to pinpoint whether it was a machine fault or a mathematical error.

What Good is a Mathematical Proof?!

So what do we have now? We seem to have said that some mathematical statements are self-evident and require no proof, while others may be "proved" but cannot be trusted! In any case, by Gödel's incompleteness theorem, mathematics can never establish its own consistency! It seems that now we have completely depleted the readers of their respect for mathematics!

Of course, what we have mentioned are some extreme cases. Proofs are still of great importance in the verification of mathematical results. The famous mathematican H. Weyl said: "Logic is the hygiene which the mathematician practices to keep his ideas healthy and strong." Another famous mathematician A. Weil said: "Rigour is to the mathematician what morality is to man."

Possibly more important than the verification purpose, *mathematical proofs provide insight and enhance understanding*. In a 1950 article entitled "The architecture of mathematics," the Bourbaki group wrote: "Indeed, every mathematician knows that a proof has not been 'understood' if one has done nothing more than verifying step by step the correctness of the deductions of which it is composed, and has not tried to gain a clear insight into the ideas which have led to the contruction of this particular chain of deductions in preference to every other one."

There is an anecdote that goes as follows: In October of 1903, the American mathematician Cole delivered a "wordless speech" to the American Mathematical Society. He wrote the following two lines on the blackboard:

$$2^{67} - 1 = 147,573,952,589,676,412,927$$

$$193,707,721 \times 761,838,257,287$$

and then proceeded to carry out the multiplication of the second line to show that the product was exactly the number on the right hand side of the first line. What he had proved was that $2^{67} - 1$ was a composite number, thus disproving a long-held belief that it was prime. Not one word was spoken, and when he put down the chalk, thunderous applause broke out. When he was asked later how long it had taken him to complete this work, his reply was "all the Sundays in the last three years." While we admire Cole for his perseverance, we feel nevertheless that his proof does not further our insight into the problem, it does not enlighten as the Russian mathematician Manin says a good proof should:

"A good proof is one which makes us wiser." It is just like the solution to Pell's equation

$$\sqrt{1+1141\times(30,693,\ldots,208)} = 1,036,\ldots,215:$$

it simply does not increase our understanding of the equation $x^2 - dy^2 = 1$.

Let me emphasize this point further by telling you Gauss' work in proving the law of quadratic reciprocity. As for the significance of this law, suffice it to say that Gauss attached so much importance to it that he called it "the gem of number theory." First, we need to know what *quadratic residue* is. Let a and m be relatively prime positive integers (i.e., they have no common factors except 1). If there is a positive integer x such that when a and x^2 are divided by m the remainders are the same, then we say that a is a quadratic residue mod m. If such an x does not exist, we say that a is not a quadratic residue mod m. For example, 3 is a quadratic residue mod 11 since the remainder of $5^2 \div 11$ is 3; but 11 is not a quadratic residue mod 3 since the remainder of $11 \div 3$ is 2 whereas the remainder of any perfect square divided by 3 is either 0 or 1.

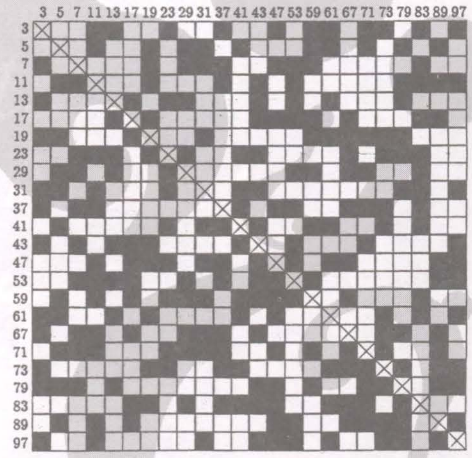


Figure 7a

Considering only a and m that are odd primes, we indicate in the following table (Figure 7a) the values of a (down) that are (are not) quadratic residues mod m (across) by a black (white) square (the square corresponding to $a = m$ are marked by X).

Now, arranging the numbers a and m by placing the primes of the form $4t + 3$ before those of the form $4t + 1$, we notice something quite curious: the table is symmetric about the diagonal except for the upper left hand corner (a and m of the form $4t + 3$ ranging from 3 to 83) which is anti-symmetric (see Figure 7b). This beautiful observation is the content of the *law of quadratic reciprocity*. Back in 1783, Euler had already mentioned a result equivalent to it, and around the same time, Legendre gave an explicit formulation of this theorem and attempted a proof. The first successful proof, due to Gauss, came only in 1796. After this, Gauss gave five more proofs of the theorem, the last one published in 1818, twenty years after the first! Through the six different proofs, Gauss revealed the different facets of the theorem, thereby deepening our understanding of number theory and illuminating the way to further research.

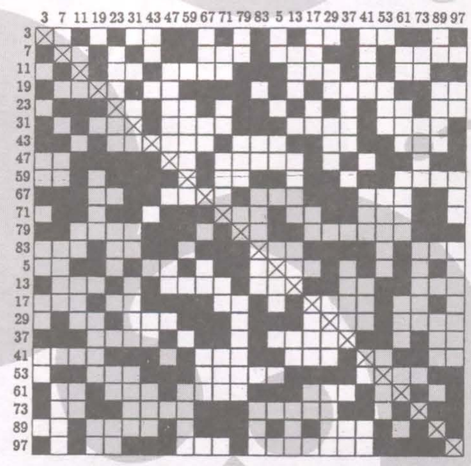


Figure 7b

In 1963, the American mathematician Gerstenhaber published a one-page paper in the *American Mathematical Monthly*, jocularly giving it the title "The 152nd proof of the law of quadratic reciprocity." More recently (in 1990), the same journal published yet another half-page paper entitled "Another proof of the quadratic reciprocity theorem?" by the American mathematician Swan. Surely, if the sole purpose of a proof is to verify, then one, or at most two, will do. Why then would the best mathematicians waste their time to give proof after proof of the same theorem?

Epilogue

In this article, we tried to provide the reader with glimpses of the human cultural activity that is mathematics. We did not, and never intended to, give any answers, hoping only that the reader would now agree that mathematics is not just a dry exercise in symbols and logic. It would be quite difficult, and indeed, possibly impossible, to define what *mathematics* is – it would depend on the individual's personal experiences. Let me end by quoting the famous mathematician and mathematics educator Polya: "Mathematical thinking is not purely 'formal'; it is not concerned only with axioms, definitions, and strict proofs, but many other things belong to it: generalizing from observed cases, inductive arguments, arguments from analogy, recognizing a mathematical concept in, or extracting it from, a concrete situation."