COMPETITION CORNER

In this issue we publish the problems of the 12th Nordic Mathematical Contest April 1998, First Japan Mathematical Olympiad, 1991, Georgian Mathematical Olympiad, May 1997, as well as the 40th International Mathematical Olympiad which was held in Bucharest, Romania, July 1999. Please send your solutions of these Olympiads to me at the address given above. All correct solutions will be acknowledged. We also present solutions of the 11th Irish Mathematical Olympiad, March 1998 and the XI Asia Pacific Mathematical Olympiad, March 1999. These were published in the last issue. Finally we also present problems and solutions of the Singapore International Mathematical Olympiad National Team Selection Test 1999/2000. The readers are urged to try the problems first before looking at the solutions.

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• Problems •

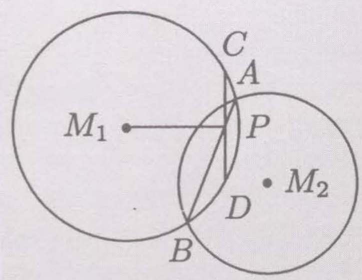
12th Nordic Mathematical Contest April 1998

1. Find all functions f from the rational numbers to the rational numbers satisfying

$$f(x+y)+f(x-y)=2f(x)+2f(y)$$

for all rational x and y.

2. Let C_1 and C_2 be two circles which intersect at points A and B. Let M_1 be the centre of C_1 and M_2 the centre of C_2. Let P be a point on the line segment AB distinct from A and B so that $|AP| \neq |BP|$. Draw the line through P perpendicular to M_1P and denote by C and D its intersections with C_1 (see figure). Similarly (not drawn in the figure), draw the line through P perpendicular to M_2P and denote by E and F its intersections with C_2. Prove that C, D, E and F are the corners of a rectangle.



3. (a) For which positive integer n does there exist a sequence $x_1, \ldots, x_n$ containing each of the numbers $1, 2, \ldots, n$ exactly once and such that k divides $x_1 + x_2 + \cdots + x_k$ for $k = 1, 2, \ldots, n$?

(b) Does there exist an infinite sequence $x_1, x_2, \ldots$ containing every positive integer exactly once and such that for any positive integer k, k divides $x_1 + x_2 + \cdots + x_k$?

4. Let n be a positive integer. Count the number of $k \in \{0, 1, \ldots, n\}$ for which $\binom{n}{k}$ is odd. Prove that this number is a power of 2, i.e., of the form 2^p for some non-negative integer p.

First Japan Mathematical Olympiad, 1991

Final Round

1. On a triangle ABC, let P, Q, R be the points which divide the segment BC, CA, AB, respectively in the ratio $t : 1-t$. Let K be the area of the triangle whose three edges have the same length with the segments AP, BQ and CR and let L be the area of triangle ABC. Find K/L in terms of t.

2. Let $\mathbb{N}$ be the set of all positive integers. The maps $p, q : \mathbb{N} \to \mathbb{N}$ are defined as follows:

$$p(1) = 2, p(2) = 3, p(3) = 4, p(4) = 1; p(n) = n \text{ if } n \geq 5.$$

$$q(1) = 3, q(2) = 4, q(3) = 2, q(4) = 1; q(n) = n \text{ if } n \geq 5.$$

(a) There exists a map $f : \mathbb{N} \to \mathbb{N}$ such that $f(f(n)) = p(n) + 2$ for all $n \in \mathbb{N}$. Find an example of such a map f.

(b) Show that there does not exist a map $g : \mathbb{N} \to \mathbb{N}$ such that $g(g(n)) = q(n) + 2$ for all $n \in \mathbb{N}$.

3. Let A be a positive integer of 16 digits in the decimal system. Prove that we can choose some successive digits from A such that their product is a square of an integer.

4. On a rectangular chess board of size 10×14, the squares are coloured white and black alternately. We write 0 or 1 in every square so that every row and every column contains an odd number of 1. Prove that the total number of 1 in the black squares is even.

5. Let A be a set of $n(\geq 2)$ points on a plane. Prove that there exists a closed circular disk with two points of A at the two ends of a diameter and which contains at least $\lfloor n/3 \rfloor$ points of A. (*Note: For any real number x, $\lfloor x \rfloor$ denotes the largest integer $\leq x$.*)

Georgian Mathematical Olympiad, May 1997

Selected problems from the final round

1. (9th Form) Prove that for any positive integer n, the following equalities hold:

$$\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor = \lfloor \sqrt{4n+3} \rfloor.$$

2. (9th Form) There are 40 participants in a mathematical competition. Each problem was marked with a '+', a '−' or '0'. After all the papers were marked it was found that no two papers had the same number of '+' and the same number of '−' marks simultaneously. What was the smallest number of problems that could have been offered to the contestants?

3. (9th Form) In the equilateral triangle ABC points D and E are chosen on the sides BC and BA, respectively, so that $\angle DAC = \angle ECA$. The lines AD and CE meet at a point F. The incircles of the triangle AFC and the quadrilateral $BDFE$ have equal radii. Find the radius of these circles if the length of the side of ABC is a.

4. (10th Form) Find all triples (x, y, z) of integers satisfying the inequality:

$$x^2 + y^2 + z^2 < xy + 3y + 2z.$$

5. (10th Form) Determine whether or not it is possible to fill an $n \times n$ table with entries equal to $1, -1$ or 0 so that when calculating the sums of the entries along the rows and the columns one could get 20 different numbers.

6. (10th Form) Prove that in any triangle, $pR \geq 2S$, where p, R, S are, respectively, the semiperimeter, circumradius and the area of the triangle.

7. (11th Form) Two positive numbers are written on a board. At each step you must perform one of the following:

(i) choose one of the numbers, say a, already written on the board and write down either a^2 or $1/a$.

(ii) choose two numbers, say a, b, already written on the board and write down either $a + b$ or $|a - b|$. How should you proceed in order that the product of the two initial numbers will eventually be written on the board?

Singapore International Mathematical Olympiad

National Team Selection Test 1999/2000. Day 1

1. In a triangle ABC, $AB > AC$, the external bisector of angle A meets the circumcircle of triangle ABC at E, and F is the foot of the perpendicular from E onto AB. Prove that $2AF = AB - AC$.

2. Find all prime numbers p such that $5^p + 12^p$ is a perfect square.

3. There are n blue points and n red points on a straight line. Prove that the sum of all distances between pairs of points of the same colour is less than or equal to the sum of all distances between pairs of points of different colours.

Day 2

4. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that for any $x, y \in \mathbb{R}$,

$$(x - y)f(x + y) - (x + y)f(x - y) = 4xy(x^2 - y^2).$$

5. In a triangle ABC, $\angle C = 60°$, D, E, F are points on the sides BC, AB, AC, respectively, and M is the intersection point of AD and BF. Suppose that $CDEF$ is a rhombus. Prove that $DF^2 = DM \cdot DA$.

6. Let n be any integer ≥ 2. Prove that $\sum 1/pq = 1/2$, where the summation is over all integers p, q which satisfy $0 < p < q \leq n$, $p + q > n$, $(p, q) = 1$.

40th International Mathematical Olympiad

Bucharest, Romania, July 1999. Day 1

1. Determine all finite sets S of at least three points in the plane which satisfy the following condition:

> for any two distinct points A and B in S, the perpendicular bisector of the line segment AB is an axis of symmetry for S.

2. Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \Big(\sum_{1 \leq i \leq n} x_i \Big)^4$$

holds for all real numbers $x_1, \ldots, x_n \geq 0$.

(b) For this constant C, determine when equality holds.

3. Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are *adjacent* if they have a common side. N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square. Determine the smallest possible value of N.

Day 2

4. Determine all pairs (p, q) of positive integers such that p is prime, $n \leq 2p$, and $(p-1)^n + 1$ is divisible by n^{p-1}.

5. Two circles Γ_1 and Γ_2 are contained inside the circle Γ, and are tangent to Γ at the distinct points M and N, respectively. Γ_1 passes through the centre of Γ_2. The line passing through the two points of intersection of Γ_1 and Γ_2 meets Γ at A and B. The lines MA and MB meet Γ_1 at C and D, respectively.

Prove that CD is tangent to Γ_2.

6. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

• Solutions •

Eleventh Irish Mathematical Olympiad, May 1998

1. Show that if x is a nonzero real number, then

$$x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0.$$

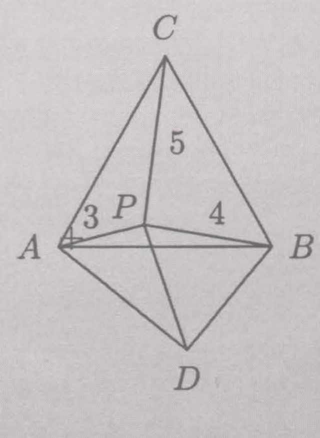
Similar solutions by Chan Sing Chun, S. Thiagarajah, Lim Chong Jie (Temasek Junior College), Kwa Chin Lum (Raffles Junior College) and Tan Chee Hau (Raffles Junior College). Also solved by Colin Tan Weiyu (Raffles Institution).

$$\begin{aligned} x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} &= \frac{x^{12} - x^9 - x^3 + 1}{x^4} \\ &= \frac{(x^6 + x^3 + 1)(x^3 - 1)^2}{x^4} \\ &= \frac{(x^3 - 1)^2\left[(x^3 + \frac{1}{2})^2 + \frac{3}{4}\right]}{x^4} \geq 0. \end{aligned}$$

2. P is a point inside an equilateral triangle such that the distances from P to the three vertices are 3, 4 and 5, respectively. Find the area of the triangle.

Solution by Chan Sing Chun. Also solved by Kwa Chin Lum, Tan Chee Hau(Raffles Junior College) and Colin Tan Weiyu (Raffles Institution).

Let the length of the side of the equilateral triangle ABC be a. Construct an equilateral triangle BPD with side equal to 4 as shown. Since $\angle ABC = \angle PBD = 60°$, we have $\angle PBC = \angle ABD$. Together with $PB = DB$, $BC = BA$, we have $\triangle PBC \equiv \triangle DBA$. Therefore $PC = DA = 5$, whence $\triangle APD$ is right angled at P and $\angle APB = 150°$. Apply the cosine rule to $\triangle APB$ we have



$$a^2 = 3^2 + 4^2 - 24\cos 150° = 25 + 12\sqrt{3}.$$

Thus the area is $(36 + 25\sqrt{3})/4$.

3. Show that no integer of the form $\overline{xyxy}$ in base 10 (where x and y are digits) can be the cube of an integer.

Find the smallest base $b > 1$ for which there is a perfect cube of the form $\overline{xyxy}$ in base b.

Similar solutions by Tan Chee Hau (Raffles Junior College), S. Thiagarajah. Also solved by Kwa Chin Lum (Raffles Junior College) and Colin Tan Weiyu (Raffles Institution).

The integer $\overline{xyxy}$ in base 10 can be represented by $1010x + 101y = 101(10x + y)$, $0 \le x, y \le 9$. Since 101 is a prime number, the number $\overline{xyxy}$ is a cube only if $10x + y$ is divisible by 101^2. But this is not possible. So $\overline{xyxy}$ cannot be a cube.

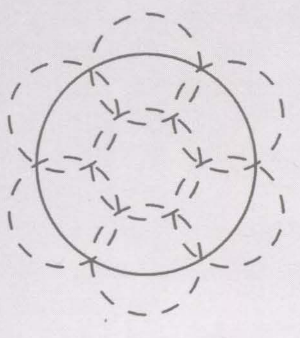
In base b, the number is $(b + b^3)x + (b^2 + 1)y = (b^2 + 1)(bx + y)$, $0 \le x, y \le b - 1$. If $b^2 + 1 = p_1 p_2 \ldots p_n$ where p_i, $i = 1, \ldots, n$ are distinct primes, then

$$\overline{xyxy} = (b^2 + 1)(bx + y) < (b^2 + 1)^3 = p_1^3 p_2^3 \cdots p_n^3.$$

Thus $\overline{xyxy}$ cannot be a cube. Therefore for $\overline{xyxy}$ to be a cube, $b^2 + 1$ must contain a square factor. By an exhaustive search the smallest b such that $b^2 + 1$ contains a square factor is $b = 7$: $b^2 + 1 = 50 = 2 \times 5^2$. We see that $50(7x + y)$ is a cube when $x = 2, y = 6$. Thus 2626 in base 7 is a cube and the smallest such b is 7.

4. Show that a disc of radius 2 can be covered by seven (possibly overlapping) discs of radius 1.

Solution by Lim Chong Jie (Temasek Junior College) and Tan Chee Hau (Raffles Junior College).



First we use 6 discs whose diameters are the sides of an inscribed regular hexagon of the disc of radius 2. Then place the seventh disc of radius 1 such that its centre coincides with that of the disc of radius 2. Then the seven discs cover the disc of radius 2.

5. If x is a real number such that $x^2 - x$ is an integer, and, for some $n \ge 3$, $x^n - x$ is also an integer, prove that x is an integer.

Solution by Kwa Chin Lum, Tan Chee Hau(Raffles Junior College). Let $x^2 - x = k \in \mathbb{Z}$. If $k = 0$, then $x = 0$ or $x = 1$ and thus x is an integer. So we suppose that $k > 0$. Then, $x^3 - x^2 = xk$ and so $x^3 = (k + 1)x + k$. In fact it is easy to prove by induction that $x^m = a_m x + b_m$ where $a_m, b_m \in \mathbb{Z}$ and that $a_m > 1$ if $m \ge 3$. Thus $x^n - x = (a_n - 1)x + b_n \in \mathbb{Z}$. It then follows that x is rational since $a_n - 1 \ne 0$. Let $x = r/s$ where r and s are coprime integers, with $s \ne 0$. Then $x^2 - x = r(r - s)/s^2 \in \mathbb{Z}$. Since r, s are coprime, we have $s \mid (r - s)$ which implies $s \mid r$, forcing $s = \pm 1$. Thus x is an integer.

6. Find all positive integers n that have exactly 16 positive integral divisors $d_1, d_2, \ldots, d_{16}$ such that

$$1 = d_1 < d_2 < \cdots < d_{16} = n, \quad \text{and} \quad d_6 = 18, d_9 - d_8 = 17.$$

Similar solutions by Tan Chee Hau (Raffles Junior College), Kwa Chin Lum (Raffles Junior College), S. Thiagarajah. Lim Chong Jie (Temasek Junior College) also provided a partial solution. Note that if $n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k}$ where $p_1, \ldots, p_k$ are distinct primes and a_i are nonnegative integers, then n has $(a_1+1)(a_2+1)\cdots(a_k+1)$ positive divisors. Thus n must be of the form p_1^{15}, $p_1p_2^7$, $p_1^3p_2^3$, $p_1p_2p_3^3$, or $p_1p_2p_3p_4$, where p_i's are distinct primes. Since 18 has six divisors: $1, 2, 3, 6, 9$, we have $d_1 = 1, d_2 = 2, d_3 = 3, d_4 = 6, d_5 = 9, d_6 = 18$ and that the first, the third the last form are ruled out. Thus we have two cases: $n = 2 \cdot 3^7; n = 2 \cdot 3^3 p$ where p is a prime larger than 18. It is easy to check that in the first case, the given conditions will not be satisfied. Thus $n = 2 \cdot 3^3 p$. We now know another two divisors of n: 27, 54. If $18 < p < 27$, then $d_7 = p$, $d_8 = 27$, $d_9 = 2p$ and $d_9 - d_8 = 17$ imply $p = 22$. Thus this is impossible. If $27 < p < 54$, then $d_7 = 27, d_8 = p, d_9 = 54$ and $d_9 - d_8 = 17$ imply $p = 37$. If $p > 54$, then $d_7 = 27, d_8 = 54, d_9 = p$. Thus $d_9 - d_8 = 17$ implies that $p = 71$. Thus the two possible values of n are $2 \cdot 3^3 \cdot 37$ and $2 \cdot 3^3 \cdot 71$.

7. Prove that if a, b, c are positive real numbers, then

$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \qquad (1)$$

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \qquad (2)$$

Similar solutions by Chan Sing Chun, Tan Chee Hau (Raffles Junior College) and Kwa Chin Lum (Raffles Junior College). Also solved by Lim Chong Jie (Temasek Junior College) and Colin Tan Weiyu (Raffles Institution). Using $AM \geq HM$ we have

$$\frac{2(a+b+c)}{3} \geq \frac{(a+b)+(b+c)+(c+a)}{3} \geq \frac{3}{\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}}.$$

Hence the first inequality follows. Using $AM \geq HM$ again, we have

$$\frac{a+b}{2} \geq \frac{2}{(1/a)+(1/b)}.$$

Thus

$$\frac{4}{a+b} \leq \frac{1}{a} + \frac{1}{b}, \quad \frac{4}{b+c} \leq \frac{1}{b} + \frac{1}{c}, \quad \frac{4}{c+a} \leq \frac{1}{c} + \frac{1}{a}.$$

The second inequality then follows.

(Note: Given n positive real numbers $a_1, \ldots, a_n$, their arithmetic mean (AM) is $(a_1+\cdots+a_n)/n$, their harmonic mean (HM) is $n/(a_1^{-1}+\cdots a_n^{-1})$. It is well known that $AM \geq HM$.)

8. Let $\mathbb{N}$ be the set of natural numbers (i.e., the positive integers).

(a) Prove that $\mathbb{N}$ can be written as a union of three mutually disjoint sets such that, if $m, n \in \mathbb{N}$ and $|m-n| = 2$ or 5, then m and n are in different sets.

(b) Prove that $\mathbb{N}$ can be written as a union of four mutually disjoint sets such that, if $m, n \in \mathbb{N}$ and $|m-n| = 2, 3$, or 5, then m and n are in different sets. Show however, that it is impossible to write $\mathbb{N}$ as a union of three mutually disjoint sets with this property.

Solution by Kwa Chin Lum (Raffles Junior College). Lim Chong Jie (Temasek Junior College) and Tan Chee Hau also obtained correct solutions to (a) and the first part of (b). (a) For each $i = 0, 1, 2$, let $A_i = \{3k + i : k \in \mathbb{N}\}$. Then A_0, A_1, A_2 have the desired properties.

(b) For each $i = 0, 1, 2, 3$, let $A_i = \{4k + i : k \in \mathbb{N}\}$. Then A_0, A_1, A_2, A_3 have the desired properties. For the second part, we prove it by contradiction. Suppose P, Q, R are three sets with the given properties. Then if $1 \in P$, we have $1+2=3, 1+3=4, 1+5=6 \notin P$. Since 4 and 6 are in different sets, we may assume that $4 \in Q$ and $6 \in R$. Also 3 and 6 are in different sets. So $3 \in Q$. We now consider 2, 5 and 7. We know 2 and 5 are in different sets, as are 2 and 7 as well as 5 and 7. Now $2 \notin Q$ since $4 \in Q$ and $5 \notin Q$ since $3 \in Q$. Thus $7 \in Q$. But 4 and 7 cannot be in the same set. Thus we have a contradiction.

9. A sequence of real numbers x_n is defined recursively as follows: x_0 and x_1 are arbitrary positive real numbers, and

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}, \quad n = 0, 1, 2, \ldots$$

Find x_{1998}.

Solution by Kwa Chin Lum (Raffles Junior College), Tan Chee Hau (Raffles Junior College), Lim Chong Jie (Temasek Junior College).

Direct computation shows that $x_2 = (1 + x_1)/x_0$, $x_3 = (1 + x_0 + x_1)/(x_0 x_1)$. After a few more terms, we have $x_5 = x_0$, $x_6 = x_1$ and so on. This suggest that $x_{n+5} = x_n$ for $n \geq 0$. This can be proved directly as follows:

$$
\begin{aligned}
x_{n+2} &= \frac{1+x_{n+1}}{x_n} \\
x_{n+3} &= \frac{1+x_{n+2}}{x_{n+1}} = \frac{1+x_n+x_{n+1}}{x_n x_{n+1}} \\
x_{n+4} &= \frac{1+x_{n+3}}{x_{n+2}} = \cdots = \frac{1+x_n}{x_{n+1}} \\
x_{n+5} &= \frac{1+x_{n+4}}{x_{n+3}} = \cdots = x_n
\end{aligned}
$$

Thus $x_{1998} = x_3 = \frac{1}{x_0 x_1}(1 + x_0 + x_1)$.

10. A triangle ABC has positive integer sides, $\angle A = 2\angle B$ and $\angle C > 90^\circ$. Find the minimum length of the perimeter.

Solution by Kwa Chin Lum (Raffles Junior College). Also solved by Tan Chee Hau (Raffles Junior College).

With the usual notation, since $\angle A = 2\angle B$, we have by the sine rule, $a/\sin A = b/\sin B$. Thus $a/b = 2\cos B$. But $\angle A + \angle B < 90^\circ$. Thus $\angle B < 30^\circ$. Hence we have $2b > a > \sqrt{3}b$ or $4b^2 > a^2 > 3b^2$. Using the fact that $A + B + C = 180^\circ$ and $A - B = B$, we also have,

$$
\begin{aligned}
& \sin(A+B)\sin(A-B) = \sin B \sin C \\
\Rightarrow\quad & \sin^2 A - \sin^2 B = \sin B \sin C \\
\Rightarrow\quad & a^2 - b^2 = bc \\
\Rightarrow\quad & c = \frac{a^2}{b} - b.
\end{aligned}
$$

(Chan Sing Chun observed that this argument is reversible and in fact we have $\angle A = 2\angle B$ if and only if $a^2 - b^2 = bc$.) Hence $b \mid a^2$. If b is a product of distinct primes, then $b \mid a^2$ implies that $b \mid a$. Thus $a \geq 2b$, a contradiction. So b is not a product of distinct primes. The perimeter is $p = a + b + c = (a^2 + ab)/b$. Thus we want to minimize p with integers a, b satisfying $4b^2 > a^2 > 3b^2$ and $b \mid a^2$. The first few integers which are not products of distinct primes are

$$4, 8, 9, 12, 16, 18, 20, \ldots.$$

The first that satisfies the required conditions is $b = 16$, $a = 28$ and $c = 33$ which give $p = 77$. This is the minimum since

$$p = (a^2 + ab)/b \geq (3b^2 + \sqrt{3}b^2)/b = (3 + \sqrt{3})b > 80 \quad \text{for} \quad b \geq 18.$$

XI Asian Pacific Mathematical Olympiad, March 1999

1. Find the smallest positive integer n with the following property: There does not exist an arithmetic progression of 1999 terms of real numbers containing exactly n integers.

Solution by Lim Yin (Victoria Junior College). Also solved by Christopher Tan Jun Yuan (Raffles Junior College). First note that integers occur at regular intervals in an arithmetic progression. Suppose in an arithmetic progression of 1999 terms, there are n integers, and between two successive integers there are d nonintegers. Suppose further that there is a total of e noninteger terms at both ends of the progression. Then we have $n+(n-1)d+e=1999$ where $0\le e\le 2d$. Therefore

$$n+\left\lfloor\frac{e-d}{d+1}\right\rfloor=\left\lfloor\frac{1999}{d+1}\right\rfloor.$$

Thus

$$n=\begin{cases}\left\lfloor\frac{1999}{d+1}\right\rfloor+1 & \text{if } e-d<0\\ \left\lfloor\frac{1999}{d+1}\right\rfloor & \text{if } e-d\ge 0\end{cases}$$

Thus we conclude that if $\lfloor\frac{1999}{d}\rfloor$ is a possible value of n then so is $\lfloor\frac{1999}{d}\rfloor+1$. Hence the answer is $\lfloor\frac{1999}{d}\rfloor+2$, where d is the largest number such that $\lfloor\frac{1999}{d-1}\rfloor-\lfloor\frac{1999}{d}\rfloor=3$. If $\lfloor 1999/d\rfloor=q$ and $\lfloor 1999/(d+1)\rfloor=q+3$, then $1999=dq+r=(d-1)(q+3)+s$, which in turn implies that $q=3(d-1)+s-r\ge 2(d-1)$. Thus $1999\ge 2d(d-1)$ and hence $d\le 32$. The sequence of values of $\lfloor\frac{1999}{d}\rfloor$ for $d=32,31,\ldots$ is $62,64,66,68,71,\ldots$. Thus the answer is 70.

2. Let a_1, a_2, ... be a sequence of real numbers satisfying $a_{i+j}\le a_i+a_j$ for all $j=1,2,\ldots$. Prove that

$$a_1+\frac{a_2}{2}+\frac{a_3}{3}+\cdots+\frac{a_n}{n}\ge a_n$$

for each positive integer n.

Official solution. Let $b_i=a_i/i$ for $i=1,2,\ldots$, we shall prove that

$$b_1+\cdots+b_n\ge a_n,\quad \text{for } n=1,2\ldots$$

by induction on n. The result is certainly true for $n=1$. Assume that

$$b_1+\cdots+b_k\ge a_k,\quad \text{for } k=1,2\ldots,n-1.$$

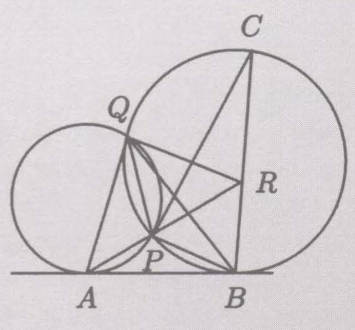
Then

$$\begin{aligned}
&nb_1 + \cdots + nb_{n-1} \\
&\quad = b_1 + (b_1 + b_2) + \cdots + (b_1 + \cdots + b_{n-1}) \\
&\qquad + (b_1 + 2b_2 + \cdots (n-1)b_{n-1}) \\
&\quad = b_1 + (b_1 + b_2) + \cdots + (b_1 + \cdots + b_{n-1}) + (a_1 + \cdots + a_{n-1}) \\
&\quad \geq 2(a_1 + \cdots + a_{n-1}) = \sum_{i=1}^{n-1} (a_i + a_{n-i}) \geq (n-1)a_n.
\end{aligned}$$

This implies $b_1 + \cdots + b_n \geq a_n$ and the proof is complete.

3. Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

Solution by He Ruimin (Raffles Junior College). Since $QPBC$ is cyclic on Γ_2, $\angle QPC = \angle QBC$. Since CP is tangent to Γ_1, $\angle QPC = \angle QAP$. Thus $\angle QAP = \angle QBC$ and we conclude that A, B, R, Q are concyclic.



Let $\angle PAB = \alpha$ and $\angle PBA = \beta$. Since AB is a common tangent to Γ_1 and Γ_2, we have $\angle AQP = \alpha$ and $\angle PQB = \beta$. Therefore, since A, B, Q, R are concyclic, $\angle ARB = \angle AQB = \alpha + \beta$ and $\angle BQR = \angle BAR = \alpha$. Thus $\angle PQR = \alpha + \beta$. Since $\angle BPR$ is an exterior angle of $\triangle ABP$, $\angle BPR = \alpha + \beta$. Thus $\angle PQR = \angle BPR = \angle BRP$. So the circumcircle of PQR is tangent to BP and BR.

4. Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

Official Solution. Without loss generality, assume that $|b| \leq |a|$. If $b = 0$, then a must be a perfect square. So $(a, b) = (k^2, 0)$ is a solution for each $k \in \mathbb{Z}$. Now consider the case $b \neq 0$. Since $a^2 + 4b$ is a perfect square, the quadratic equation

$$x^2 + ax - b = 0 \qquad (*)$$

has two nonzero integral solutions x_1, x_2 with $|x_1| \leq |x_2|$. We have $x_1 + x_2 = -a$ and $x_1 x_2 = -b$. From these we have

$$\frac{1}{|x_1|} + \frac{1}{|x_2|} \geq \left| \frac{1}{x_1} + \frac{1}{x_2} \right| = \frac{|a|}{|b|} \geq 1.$$

Hence $|x_1| \leq 2$.

(1) $x_1 = 2$: Substituting $x_1 = 2$ into (*), we have $b = 2a + 4$. So $b^2 + 4a = (2a+4)^2 + 4a = (2a+5)^2 - 9$. If is easy to see that the solution in nonnegative integers of the equation $x^2 - 9 = y^2$ is $(3, 0)$. Hence $2a + 5 = \pm 3$. From this we get $(-4, -4)$ and $(-1, 2)$ with the latter discarded because of the condition $|a| \geq |b|$.

(2) $x_1 = -2$: Substitution gives $b = 4 - 2a$. Hence, $b^2 + 4a = (2a-3)^2 + 7$. The nonnegative integer solution of $x^2 + 7 = y^2$ is $(3, 4)$. Thus $2a - 3 = \pm 3$. From this we get the solution $(3, -2)$.

(3) $x_1 = 1$: Substitution yields $b = a + 1$. Hence $b^2 + 4a = (a+3)^2 - 8$. Proceeding as before, we get the solution $(-6, -5)$.

(4) $x_1 = -1$: Substitution yields $b = 1 - a$. Thus $a^2 + 4b = (a-2)^2$ and $b^2 + 4a = (a+1)^2$. Consequently, we get the solutions $(k, 1-k)$, $k \in \mathbb{Z}$.

Solution by Tay Kah Keng (Raffles Junior College) with gaps filled by the editor. Suppose that $a^2 + 4b = m^2$ and $b^2 + 4a = n^2$ where m and n are nonnegative integers. We have $a \equiv m$ and $b \equiv n \pmod 2$. Thus we can write $m = a + 2x$ and $n = b + 2y$ for some integers x, y. We also have $a = by + y^2$ and $b = ax + x^2$.

(1) $a, b > 0$: Here x, y are both positive integers. Since $a^2 + 4b = m^2 = (a+2x)^2$, we have $b = x(a+x) > a$. Similarly, $a = y(b+y) > b$ and we have contradiction.

(2) $a = 0$ or $b = 0$: When $a = 0$, b is a perfect square and when $b = 0$, a must be perfect square. Thus $(a, b) = (0, d^2), (d^2, 0), d \in \mathbb{Z}$, are both solutions.

(3) $a, b < 0$: We assume without loss of generality that $a \geq b$. Then $a^2 + 4b \geq 0$ implies $a^2 \geq -4b \geq -4a$. Thus $a \leq -4$. When $a = -4$, $b = -4$. When $a = -5$, $b = -5, -6$. When $a = -6$, $-6 \geq b \geq -9$. Of these only $(a, b) = (-4, -4), (-5, -6)$ are solutions. Now consider the case $a \leq -7$. From $b^2 + 4a = n^2 = (b+2y)^2$, we have $y \geq 1$. If $y = 1$, then $b = a - 1 \leq -8$. Thus $a^2 + 4b = b^2 - 6|b| + 1$ and this is not a square since $(|b|-4)^2 \leq b^2 - 6|b| + 1 \leq (|b|-3)^2$. Thus $y \geq 2$ or $b \leq a - 2$. But

$$b^2 + 4b \leq b^2 + 4a = (b+2y)^2 \leq (b+4)^2, \quad \text{or} \quad b \geq -4$$

which is impossible.

(4) $a > 0, b < 0$: From $b^2 + 4a = (b+2y)^2$, and $a^2 + 4b = (a+2x)^2$, we have $x, y \leq -1$. If $y = -1$ or $x = -1$, then $a = 1 - b$ and indeed $(a, b) = (k, 1-k)$ satisfies both equations. If $x, y \leq -2$, then $a = |y|(|y| + |b|) \geq 2|b|$ and $|b| = |x|(a - |x|) \geq |x|$. Thus

$$a \geq 2|x| \Rightarrow |b| \geq |x|^2 \Rightarrow a \geq 2|x|^2 \Rightarrow |b| \geq |x|^3 \ldots$$

Since $|x| \geq 2$, we see that there is no solution.

Combining the four cases, we have the following solutions:

$$(-4,-4),(-5,-6),(-6,-5),(0,k^2),(k^2,0),(k,1-k) \text{ where } k \in \mathbb{Z}.$$

5. Let S be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of S on its circumference, $n-1$ points in its interior and $n-1$ in its exterior. Prove that the number of good circles has the same parity as n.

Solution by the editor. For any two points A and B, let P_1, P_2, ..., P_k be points on one side of the line AB and P_{k+1}, ..., P_{2n-1} be points on the other side. We shall prove that the number of good circles passing through A and B is odd. Let

$$\theta_i = \begin{cases} \angle AP_iB & \text{if } i=1,\ldots,k \\ 180^\circ - \angle AP_iB & \text{if } i=k+1,\ldots,2n-1 \end{cases}$$

It is easy to see that P_j is in the interior of the circle ABP_i, if and only if

$$\begin{cases} \theta_j > \theta_i & \text{for } 1 \le j \le k \\ \theta_j < \theta_i & \text{for } k+1 \le j \le 2n-1 \end{cases}.$$

Arrange the points P_i in increasing order of their corresponding angles θ_i. Colour the points P_i, $i=1,\ldots,k$, black and the points P_i, $i=k+1,\ldots,2n-1$, white. For any point X (different from A and B), let B_X be the number of black points less than X minus the number of black points greater than X and W_X be the corresponding difference for white points. (Note that black points which are greater than X are interior points of the circle ABX while the white points greater than X are exterior points.) Define $D_X = B_X - W_X$. From the forgoing discussion we know that $\triangle ABX$ is good if and only if $D_X = 0$. We call such a point *good*. If $X < Y$ are consecutive points, then $D_X = D_Y$ if X and Y are of different colours. (It is easy to show that $D_Y - D_X = -2$ if X and Y are both white and $D_Y - D_X = 2$ if X and Y are both black. But we do not need these.)

If all the points are of the same colour, there is only one good point, namely the middle point among the P_i's.

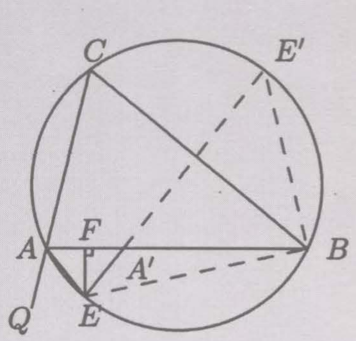
Now we suppose that there are points of either colour. Then there is a pair of adjacent points, say X, Y, with different colour. Since $D_X = D_Y$, either both are good or both are not good. Their removal also does not change the value of D_Z for any other point Z. Thus the removal of a pair of adjacent points of different colour does not change the parity of the number of good points. Continue to remove such pairs until only points of the same colour are left. When this happens there is only one good point. Thus the number of good circles through A and B is odd.

Now let g_{AB} be the number of good circles through A and B. Since each good circle contains exactly three points, i.e., three pairs of points. Then $\sum g_{AB} = 3g$ where g is total number of good circles. Since there are a total of $n(2n+1)$ terms in the sum, and each term is odd, we have $g \equiv n \pmod 2$.

Singapore International Mathematical Olympiad

National Team Selection Test 1999/2000

1. In a triangle ABC, $AB > AC$, the external bisector of angle A meets the circumcircle of triangle ABC at E, and F is the foot of the perpendicular from E onto AB. Prove that $2AF = AB - AC$.



Official solution. Let A' be the point on AB such that $A'F = FA$. Then $\triangle AEA'$ is isosceles. Extend EA' meeting the circumcircle of $\triangle ABC$ at E'. Join BE' and BE. Since $\angle ABC = \angle EBC - \angle ABE = \angle AA'E - \angle ABE = \angle E'EB$, we have $BE' = AC$. Also, $\triangle AEA'$ is similar to $\triangle E'BA'$ implies that $A'B = BE' = AC$. Hence, $2AF = AB - A'B = AB - AC$.

(*Remark:* Let PA be the tangent at A with P inside the sector of $\angle QAE$. As $AB > AC$, we have $\angle C > \angle B$. Hence, $\angle PAB = \angle C > \angle B = \angle QAP$. This implies that E is on the arc AB not containing C. Also, $\angle EBF = \angle PAE < \angle EAB$ so that $BF > AF$. Hence, A' is between F and B.)

2. Find all prime integers p such that $5^p + 12^p$ is a perfect square.

Official solution. The problem can be changed to find all integers m such that $5^m + 12^m$ is a perfect square. Again the only answer is $m = 2$. We shall give the solution in this more general case. (The solution of the original problem is easy by considering mod 5 or mod 10.)

One solution is $p = 2$ and we assert that it is the only solution. If $p = 2k+1$ is odd, then $5^{2k+1} + 12^{2k+1} \equiv 2^{2k+1} \equiv 2 \cdot 4^k \equiv 2(-1)^k \equiv 2$ or $3 \pmod 5$. However the square of an integer can only be $0, 1$ or $4 \pmod 5$. So $5^p + 12^p$ is not a square when p is odd.

Now suppose that $5^{2n} + 12^{2n} = t^2$ with $n \geq 2$. Then

$$5^{2n} = t^2 - 12^{2n} = (t - 12^n)(t + 12^n).$$

If 5 divides both factors on the right, it must also divide their difference which means it divides 12. But this is impossible. Thus $t - 12^n = 1$ and

$$5^{2n} = 2 \cdot 12^n + 1 \quad \text{or} \quad 2^{2n+1}3^n = (5^n - 1)(5^n + 1).$$

If n is odd, then $3 \mid 5^n + 1$ and $3 \nmid 5^n - 1$. Thus $5^n + 1 = 2 \cdot 3^n$ and $5^n - 1 = 4^n$ which cannot hold for $n > 1$. If n is even, then $5^n - 1 = 2 \cdot 3^n$ and $5^n + 1 = 4^n$, which again cannot hold for $n \geq 2$. Thus there is no solution for $p = 2n$, $n \geq 2$.

3. There are n blue points and n red points on a straight line. Prove that the sum of all distances between pairs of points of the same colour is less than or equal to the sum of all distances between pairs of points of different colours.

Solution by Tan Chee Hau (Raffles Junior College). We shall prove the assertion using induction on n. Let $x_1, x_2, \ldots, x_n$ be the coordinates of the n red points on the real line. Similarly, let $y_1, y_2, \ldots, y_n$ be the coordinates of the n blue points on the real line. Let S_n be the sum of distances of points of the same colour, D_n the sum of distances of points of different colours. If $n = 1$, then $S_1 = 0$ and $D_1 = |x_1 - y_1|$. Clearly, $D_1 \geq S_1$. Now suppose $D_{n-1} \geq S_{n-1}$.

$$S_n - S_{n-1} = \sum_{i=1}^{n} (x_n - x_i) + (y_n - y_i) = \sum_{i=1}^{n} (x_n - y_i) + (y_n - x_i)$$

$$D_n - D_{n-1} = |x_n - y_n| + \sum_{i=1}^{n} |x_n - y_i| + |y_n - x_i| \geq S_n - S_{n-1}.$$

It follows from this and the induction hypothesis that $D_n \geq S_n$.

Solution by Lim Yin. Take 2 consecutive points A and B with the coordinate of A less than the coordinate of B. Suppose that there are k blue points and l red points with their coordinates less than or equal to the coordinate of A. Then the segment AB is covered $(n-k)k + (n-l)l$ times by segments whose endpoints have the same colour, and $(n-k)l + (n-l)k$ times by segments whose endpoints have different colours. Since $(n-k)k + (n-l)l \leq (n-k)l + (n-l)k$, the assertion follows by summing the lengths of all these segments over all pairs of consecutive points.

Solution by Julius Poh. Let S be the total length of the segments whose endpoints are of the same colour and D be the total length of the segments whose endpoints are of different colour. Move the leftmost point to the right by a distance x. Then S decreases by $(n-1)x$ while D decreases by nx. Thus D decreases more than S. Continue to move this point until it hits the next point. If these two points are of different colour, then deleting them causes S and D to decrease by the same amount. If they are of the same colour, then continue to move the pair to the right and in the process D decreases more than S does. We continuing this process, when the block that we are moving (all points in the block are of the same colour) hits a point which is of different colour, remove a pair of points of different colour. If it hits a point of the same colour, then add the point to the block and continue moving to the right. Eventually all the points will be removed and both S and D have decreased to 0. Thus at the beginning $D \geq S$.

4. Find all functions $f : \mathbb{R} \longrightarrow \mathbb{R}$ such for any $x, y \in \mathbb{R}$,

$$(x-y)f(x+y)-(x+y)f(x-y)=4xy(x^2-y^2).$$

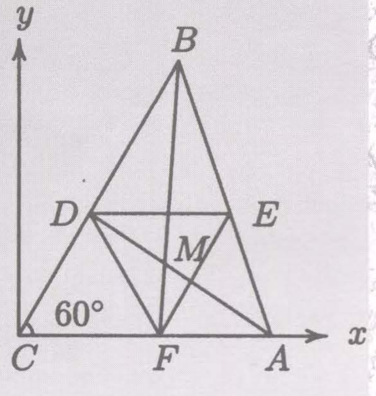
Official solution. Let $x = y = 1$. We have $f(0) = 0$. Let $a = x + y$ and $b = x - y$. Then the given functional equation is equivalent to $bf(a) - af(b) = (a^2 - b^2)ab$. This holds for all real numbers a and b. For nonzero a and b, this can be rewritten as

$$\frac{f(a)}{a}-a^2=\frac{f(b)}{b}-b^2.$$

Hence, for any nonzero real number x, $\frac{f(x)}{x} - x^2 = f(1) - 1$. Let $\alpha = f(1) - 1$. We have $f(x) = x^3 + \alpha x$, for all $x \neq 0$. As $f(0) = 0$, we thus have $f(x) = x^3 + \alpha x$ for all $x \in \mathbb{R}$. Clearly $f(x) = x^3 + \alpha x$ satisfies the given relation.

5. In a triangle ABC, $\angle C = 60°$, D, E, F are points on the sides BC, AB, AC respectively, and M is the intersection point of AD and BF. Suppose that $CDEF$ is a rhombus. Prove that $DF^2 = DM \cdot DA$.

Official solution. Set up a coordinate system with CA on the x-axis and $C = (0,0)$. Let $A = (a, 0)$ with $a > 0$, $F = (1,0), D = (1/2, \sqrt{3}/2)$, and $E = (3/2, \sqrt{3}/2)$. Then,



$$B=\left(\frac{a}{2(a-1)},\frac{\sqrt{3}a}{2(a-1)}\right),$$
$$M=\left(\frac{a(1+a)}{2(1-a+a^2)},\frac{\sqrt{3}a(a-1)}{2(1-a+a^2)}\right).$$

Hence, $DF = 1, DA^2 = (\frac{1}{2} - a)^2 + \frac{3}{4} = 1 - a + a^2$, and

$$DM^2=\left(\frac{a(1+a)}{2(1-a+a^2)}-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}a(a-1)}{2(1-a+a^2)}-\frac{\sqrt{3}}{2}\right)^2=\frac{1}{1-a+a^2}.$$

Solution by Tay Kah Keng (Raffles Junior College). Since DE is parallel to CA, ΔDEB is similar to ΔFAE so that $DB : DE = FE : FA$. As $CDEF$ is a rhombus, we have $DE = FE = DF$. Hence, $DB : DF = FD : FA$. Also, $\angle BDF = \angle DFA = 120°$. This shows that ΔBDF is similar to ΔDFA. Therefore, $\angle DFB = \angle FAD$. This implies that ΔDMF is similar to ΔDFA. Consequently, $DF^2 = DM \cdot DA$.

6. Let n be any integer ≥ 2. Prove that $\sum 1/pq = 1/2$, where the summation is over all integers p, q which satisfy $0 < p < q \leq n$, $p+q > n$, $(p,q) = 1$.

Official solution. Let $f(n)$ be the given sum. The summands that appear in $f(n)$ but not in $f(n-1)$ are those of the form $a_p = 1/pn$ where $1 \leq p < n$, $(p,n) = 1$; the summands in $f(n-1)$ but not in $f(n)$ are those of the form $b_p = 1/p(n-p)$ where $1 \leq p < n-p$, $(p, n-p) = 1$, equivalently $(p,n) = 1$. (For example, if $n = 10$, those summands in $f(10)$ but not in $f(9)$ are $\frac{1}{1\times 10}, \frac{1}{3\times 10}, \frac{1}{7\times 10}, \frac{1}{9\times 10}$, while those which are in $f(9)$ but not in $f(10)$ are $\frac{1}{1\times 9}, \frac{1}{3\times 7}$.) Hence summing only over values of p such that $(p,n) = 1$, we have

$$f(n) - f(n-1) = \sum_{p<n} a_p - \sum_{2p<n} b_p = \sum_{2p<n} (a_p + a_{n-p} - b_p).$$

But $a_p + a_{n-p} - b_p = 0$; hence $f(n) = f(n-1)$ for all $n \geq 3$, and the result follows.