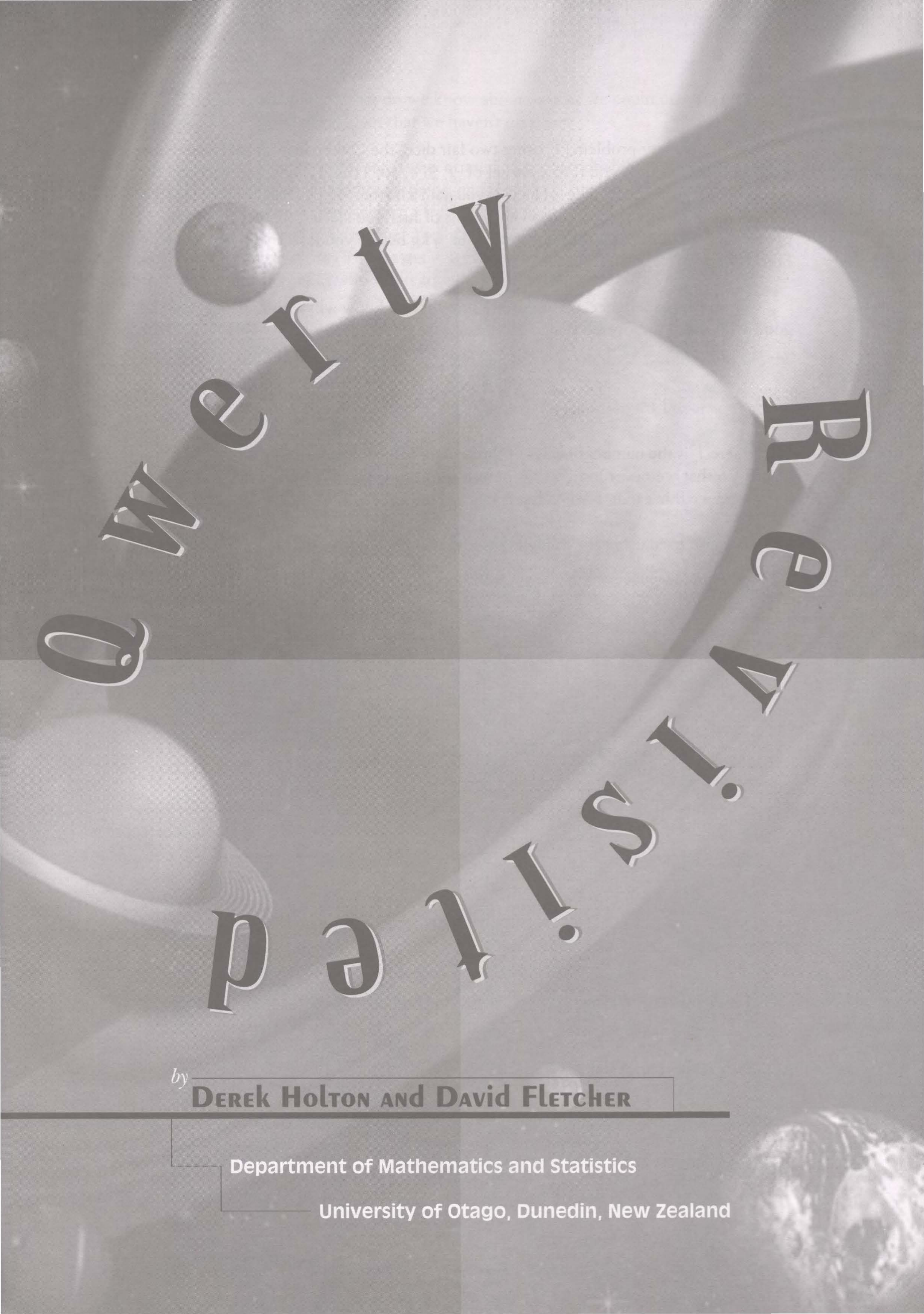

Qwerty Revisited

by

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Abstract

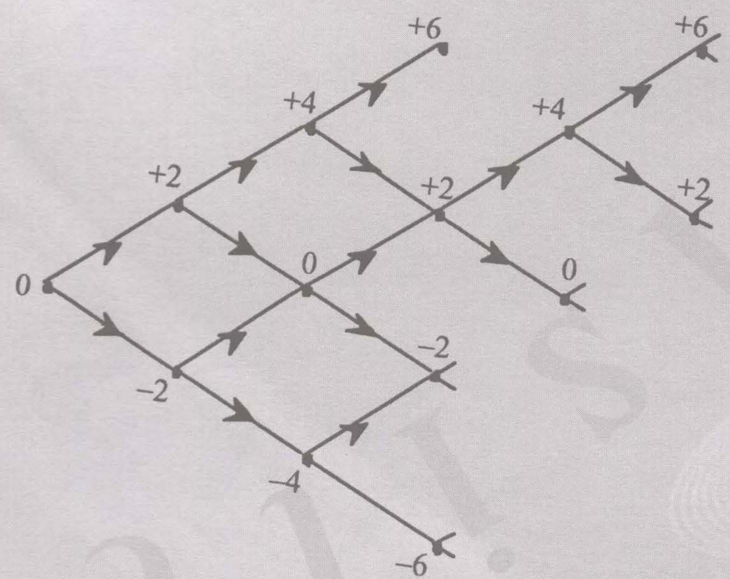
In the Qwerty problem [1] using two fair dice, the Qwertians will give you 2 kg of super neutron fuel each time you throw a total of 2, 3, 4, 10, 11 or 12 and you will give them 2 kg otherwise. You start with 6 kg of fuel. If you gain a further 6 kg of fuel, your space ship can just get you back to Earth. If you lose the 6 kg of fuel you started with, you will be a slave on Qwerty for 10,000 years. But you must gain 6 kg before you lose 6 kg! What is your chance of freedom?

From past experience, we are pretty sure that we know what your chance p is (and it ain't good). We have discovered that

$$p = \frac{1}{27}\sum_{n=0}^{\infty} f_n \left(\frac{2}{9}\right)^n$$

Here f_n is the number of ways of throwing a sequence of wins (+) and losses (-)* of 2 kgs of fuel, so that we never lose 6 kgs before we gain 6 kg. We know that $f_0 = 1$, $f_1 = 3$ and $f_2 = 9$ and we think that $f_n = 3^n$. But is it?

We also tried using the following tree diagram. It turned out to be of some use.



Now, as they say, read on.

Mind Your Ps And Qs

We've been hell bent on trying to find the wretched f_n. Maybe we can go round it somehow. Maybe we don't have to find a specific expression for f_n (like 3^n). Perhaps we can creep up on p via the back door.

* We show how to get f_2 on p.77.

So how could we do that? What do we know about p that we could use? What do we know about the whole Qwerty problem that we haven't used yet?

When you think about it, we've only concentrated on p, the probability of escaping. We've given no thought to q, the probability of being enslaved for a small period of 10 millennia. Maybe q is actually easier to find than p. If it is then, as they say, we're laughing. Clearly $p + q = 1$. You can only escape or be a slave. You may be able to stave off the exit hour by several million throws of the dice but they'll either get you eventually or you'll escape. (Of course, we're assuming you won't die first. But who wants to worry about that possibility?)

Naturally we can find q using a system of equations as we did with p. That will clearly give us $q = \frac{8}{9}$ but that doesn't give us any new insight into the problem (It might be a useful exercise though to check that $q = \frac{8}{9}$ and so practice the process shown in the last article.) How else could we get q?

Hmmm. We know this doesn't help much but $q = \sum_{n=0}^{\infty} g_n \left(\frac{1}{3}\right)^a \left(\frac{2}{3}\right)^b$, we mean it's certain that q has the same form as p when expressed as an infinite sum. Of course, in this case g_n is the number of ways of getting to a -6 situation by "throwing" pluses and minuses . Here, instead of there being n minuses, there are n pluses. So here there must be three more minuses than pluses. That means $a = n$ and $b = n + 3$, which gives

$$\begin{aligned} q &= \sum_{n=0}^{\infty} g_n \left(\frac{1}{3}\right)^n \left(\frac{2}{3}\right)^{n+3} \\ &= \frac{8}{27} \sum_{n=0}^{\infty} g_n \left(\frac{2}{9}\right)^n \end{aligned}$$

So what is g_n? Now that's the wrong question. If we could find g_n directly, we'd be able to find f_n directly and we wouldn't be in this pickle in the first place. OK, think. How can we find g_n without finding g_n? Is there some link between g_n and f_n (like the link between p and q, $p + q = 1$), that we can exploit? What is g_n? What is f_n? What?

The tree diagram was useful before (refer [1]). Maybe we can use it again. Now f_n could be thought of as the number of ways of getting to a "good" node, having "lost" the toss n times. In the same way, g_n is the number of ways of getting to a "bad" node, having "won" the toss n times. Hey ... Surely not ... Could g_n be equal to f_n? Is that possible? If it is, then there is a good reason for it.

Symmetry!? Isn't it all symmetric? Suppose we've got to a good node using n minuses. What happens if we change all the pluses to minuses and all the minuses to pluses? Surely that shows us how to get to a bad node. And vice-versa. For every way of getting to a bad node, there's a mirror-image way of getting to a good one.

Fantastic! So $f_n = g_n$! So what? It's nice to know, but how can we use it? What is it that we've actually got?

Let's recap for a minute. We know that

$$p = \frac{1}{27}\sum_{n=0}^{\infty} f_n \left(\frac{2}{9}\right)^n$$

and $$q = \frac{8}{27}\sum_{n=0}^{\infty} g_n \left(\frac{2}{9}\right)^n$$

and $f_n = g_n$.

So $$q = \frac{8}{27}\sum_{n=0}^{\infty} f_n \left(\frac{2}{9}\right)^n.$$

Ah but then, surely,

$$q = 8p?$$

Yes, that looks good to us.

Ah but now we're in with a show because we know that

$$p + q = 1$$

and $q = 8p$.

Naturally then

$$q = 8p = 1,$$

so $p = \frac{1}{9}$. Bingo!

But, but...

It would be easy to give up at this stage. We've solved the Qwertian problem in two ways already, maybe three, or four. But we still don't know for sure that $f_n = 3^n$. All that garbage at the end of the last article [1] about two infinite sums is garbage. Clearly, well maybe not, but it ain't true anyway, it isn't the case that if $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} b_n$, then $a_n = b_n$.

It's easy to see if you think about it. Let $a_{2n} = b_{2n+1}$ and $a_{2n+1} = b_{2n}$. That should do it. We apologise for even bringing it up. But we still don't have an explicit value for f_n. What to do?

Qwerty Revisited

Let's go back and look at some examples. This is always a good way to start. With n = 2 we have the following nine cases

Throw						
1st	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**
−	−	+	+	+	+	+
−	+	−	+	+	+	+
−	+	+	−	+	+	+
−	+	+	+	−	+	+
+	−	−	+	+	+	+
+	−	+	−	+	+	+
+	−	+	+	−	+	+
+	+	−	−	+	+	+
+	+	−	+	−	+	+

Can we tell anything from that? Probably not. Why don't you go and work out f_3? The rules are that there are three minuses, that the sum of all pluses and minuses is +3, and that never before the end do we get a partial sum of +3 or -3. So we can't include any of the following in our tally for f_3:

Throw								
1st	**2nd**	**3rd**	**4th**	**5th**	**6th**	**7th**	**8th**	**9th**
+	+	+	−	−	−	+	+	+
−	+	+	−	+	+	+	−	+
−	−	−	+	+	+	+	+	+

So what is f_3? We've predicted that $f_3 = 3^3 = 27$ and $f_4 = 3^4 = 81$. Is that what you've found? If not, spend a little time and effort and see what you get. (For the lazy ones amongst you we'll move right along. For the others, we'll still be here when you get back.)

So how did you go? Turn out the way we expected? Did you learn anything? (You probably learned that being systematic was a great idea.) Have you done enough to work out what f_n might be?

One thing that seems to be happening is that we always end with a + + . Does that have to be the case? If so, why? If not, why not?

OK but suppose we stick to the conjecture that $f_n = 3^n$. How could we prove something like this? How does 3^n come up? It seems to us that there are at least two ways. You can get 3^n because there are three objects, any of which can go into n places. You can also get 3^n by showing that $f_n = 3f_{n-1}$. This is because

$$f_n = 3f_{n-1} = 3(3f_{n-2}) = 3^2(3f_{n-3}) = \dots = 3^n f_0 = 3^n.$$

Which one of these is worth trying? How can we put three objects into n places? What would the three objects be? We only seem to have worked with two things - pluses and minuses. Fair enough. So how can we show that $f_n = 3f_{n-1}$? That's of course, assuming that it is.

It's always worth trying things out on an example first. All the nine arrangements above that make up f_2 end in $++$. That has to be the case. If they ended in anything else the partial sum would have been $+3$ earlier on in the sequence. Just before the last $++$ there seem to be three other paired possibilities. They are $+-$, $-+$ and $++$. Why no $--$? We think it's clear that if a sequence ended in $--++$, then $+3$ would have happened earlier.

So let's pull out the $+-$ and see what we have left.

$$\begin{array}{cccccccccccc}
- & + & + & (+-) & + & + & \longrightarrow & - & + & + & + & + \\
+ & - & + & (+-) & + & + & \longrightarrow & + & - & + & + & + \\
+ & + & - & (+-) & + & + & \longrightarrow & + & + & - & + & +
\end{array}$$

the interesting thing here is that removing $+-$ leads to the three possibilities for f_1. Does the same thing happen for $-+$?

$$\begin{array}{cccccccccccc}
- & + & + & (-+) & + & + & \longrightarrow & - & + & + & + & + \\
+ & - & + & (-+) & + & + & \longrightarrow & + & - & + & + & + \\
+ & + & - & (-+) & + & + & \longrightarrow & + & + & - & + & +
\end{array}$$

It worked again. It looks as if we may have a way to go from f_2 to f_1 or vice-versa. Ah wait though. Deleting $++$ isn't going to take us down a step. To go from f_2 to f_1 we need to remove a minus sign. Let's do it anyway and see what we get

$$\begin{array}{cccccccccccc}
- & - & + & (++) & + & + & \longrightarrow & - & - & + & + & + \\
- & + & - & (++) & + & + & \longrightarrow & - & + & - & + & + \\
+ & - & - & (++) & + & + & \longrightarrow & + & - & - & + & +
\end{array}$$

Well, there are certainly three reduced sequences here but they aren't f_1! They would be though, if we interchanged pluses and minuses in the first three terms.

That all sounds a bit loopy. Will it always work? First will inserting a $+-$ or $-+$ into a proper arrangement with $n - 1$ minuses always give us a proper n arrangement? Let's start with the $+-$ and see what happens.

Suppose we have a proper sequence of $n-1$ minuses and $n+2$ pluses. We know that at no stage do the partial sums add to $+3$ or -3, except at the end when they are $+3$. We also know that the sequence finishes with $++$. So just before the $++$, the sum is precisely $+1$. If we insert $+-$ before the end $++$, then the final partial sums are $+2, +1, +2, +3$. So we **do** produce a proper arrangement with n minuses.

[..................] + +	$\longrightarrow$	[..................] (+ −) + +
proper sequence with $n-1$ minuses	insert (+ −)	proper sequence with n minuses

So for every $(n-1)$–minuses sequence we can get an n–minuses sequence. Oh and vice versa. The arrow in the above diagram can go back the other way. Hence we have an equivalence between $n-1$ sequences and n sequences with $+-$ next to the end $++$. This means there must be f_{n-1} sequences with 'n' minuses which end $\ldots +-++$.

But exactly the same argument can be applied to sequences with n minuses which end ... $- + + +$. So there are f_{n-1} of them too.

Now all we need to be able to do is to master the dodgy argument of the ... $+ + + +$ sequences. We're a bit wary about interchanging the $-$ and $+$ signs. Let's take a deep breath and give it a go.

[ ] + +	$\longrightarrow$	[ ]* + +	$\longrightarrow$	[ ]* + + + +
sequence with n - 1 minuses and n pluses in the square bracket	Swap + and –	n - 1 pluses and and n minuses in in the square bracket		$n + 3$ pluses altogether; n minuses altogether

The diagram above tells us what we have to do. It also shows that we can go via this fiddle, from an $(n-1)$–minuses sequence to an n–minuses sequence, so **that** part of the book-keeping is OK. Does everything else work out? Do we ever get $+3$ or -3 before the end of the n–minuses sequence?

Look at the square bracket. No partial sums in that reach $+3$ or -3 and it ends with a sum of $+1$ so that the final $+ +$ give a total of $+3$. So in the square bracket with an asterisk, the partial sums nowhere reach -3 or $+3$ and the final sum is -1. (By interchanging $+$ and $-$ we just interchange the sign of the sums.) When we finally add $+ + + +$ to the end we have no $+3$ or -3 in the asterisked square bracket and the final five sums are $-1, 0, 1, 2, 3$. So we do produce a **good** arrangement with n minuses.

We hope it's now clear that for every $(n-1)$–minuses sequence, interchanging appropriate pluses and minuses and inserting $+ +$ gives us an n –minuses sequence, and vice versa. So there are as many n -minuses sequences ending $+ + + +$ as there are $(n-1)$–minuses sequences. And that's f_{n-1}.

Putting $+ -, - +$ and $+ +$ together we've proved our conjecture. It's clear that f_n really does equal $3f_{n-1}$. So $f_n = 3^n$. At last we've justified it. But is there an easier way?

Footnote
This whole problem is part of quite a large piece of literature in an area called Gambler's Ruin. People have long been interested in winning large amounts of money and by continually betting against an adversary, perhaps doubling the stakes as they go, is one way of trying to achieve this. Consequently, mathematicians have had great pleasure in analyzing such situations. A readable introduction can be found in The Theory of Stochastic Processes by D.R. Cox and H.D. Miller (1965), while if you're feeling adventurous you could delve into the delights of a detailed analysis of this problem in Chapter XIV of William Feller's excellent An Introduction to Probability Theory and Its Applications (Third Edition, 1968, Wiley).

Without looking at a reference though see if you can handle the Qwertian problem if we remove the symmetry. How easy is it to escape if you only need a further 4kg of super neutron fuel instead of the 6 kg above?

Reference
[1] D. Holton and D. Fletcher, The Qwertians Ten Thousand Years of Slavery, Mathematical Medley, Volume 26 No. 1 (1999), 9-13.