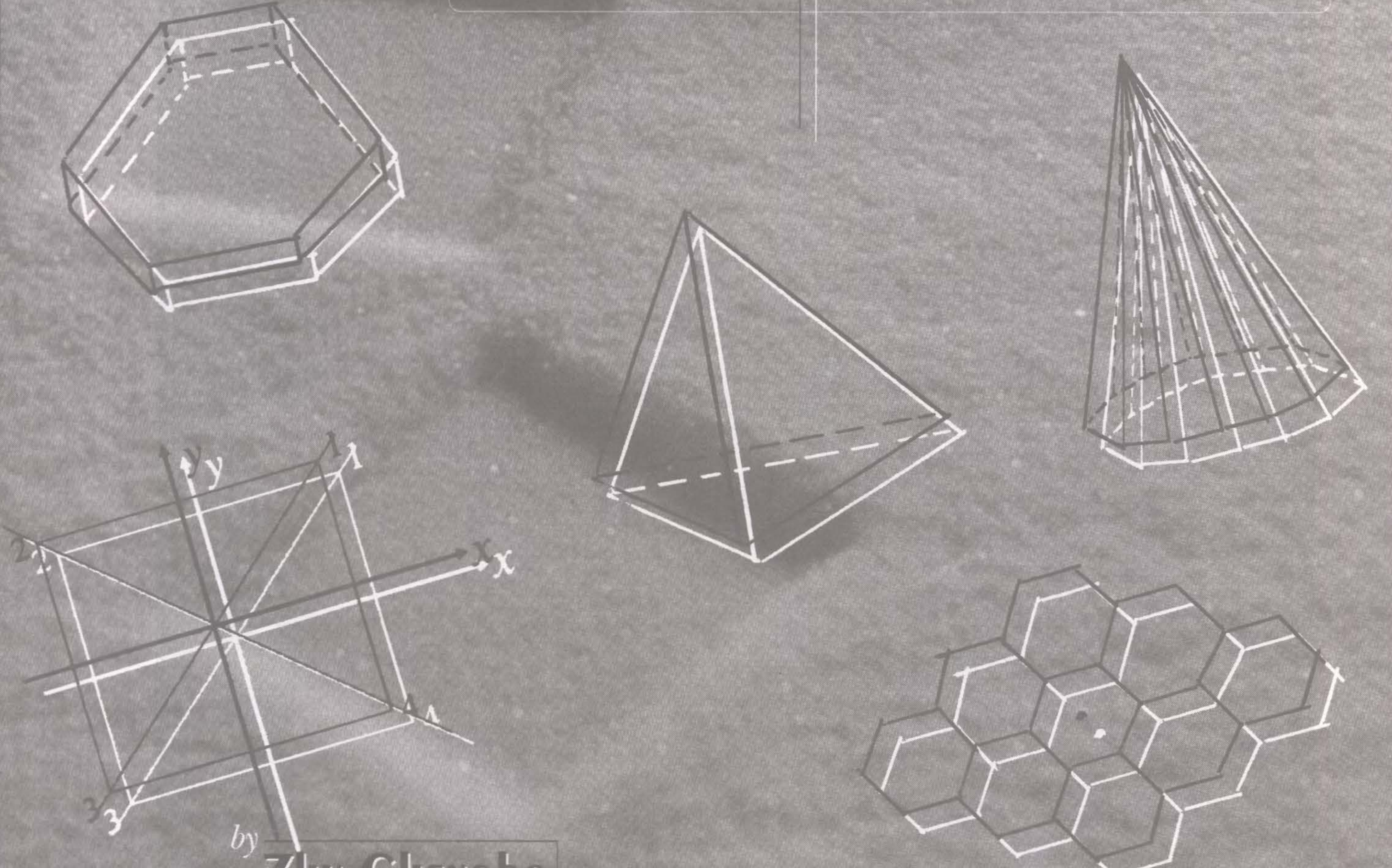
Groups and Symmetry



by

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1 Groups Measure Symmetry

Consider a square in the xy-plane with its center at the origin and its sides parallel to the x and y-axes (Figure 1).

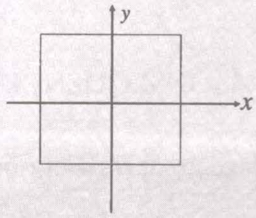


Figure 1

There are eight transformations of the plane, which leave the square in its original position. These transformations, four of which are rotations and four of which are reflections, are called the symmetries of the square.

r_0	rotation through zero.
r_1	rotation through $\pi / 2$.
r_2	rotation through π.
r_3	rotation through $3\pi / 2$.
s_1	reflection in the x-axis.
s_2	reflection in the $y = x$ diagonal.
s_3	reflection in the y-axis.
s_4	reflection in the $y = -x$ diagonal .

We say that these eight symmetries form a group, which we denote by D_4. This means in particular that the eight elements can be combined together. If f and g are any two of these symmetries then there is a symmetry, which we denote by fg, that has the same effect as the combined effect of first applying g then f. For example, consider $r_1 s_3$. To see how the composition affects the square, label the vertices 1, 2, 3 and 4 as shown. Then apply the transformations in turn, remembering to apply s_3 first, then r_1. You should find that 1 ends up at the bottom left and 2 ends up where it started, at top left. The overall effect is the same as that achieved by s_4, in other words, $r_1 s_3 = s_4$.

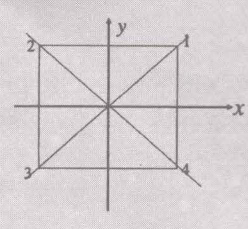


Figure 2

D_4 is one of an infinite family of so-called dihedral groups denoted by D_n which is the group of symmetries of a regular n-sided polygon. As in the case of D_4, there are two kinds of symmetries in D_n, half of those are rotations, and the other half are reflections. The rotations are by the angles $2k\pi / n$, where $k = 0, 1, 2, ..., n-1$. They themselves form a subgroup, meaning that they can be combined among themselves. It is called a cyclic group of order n.

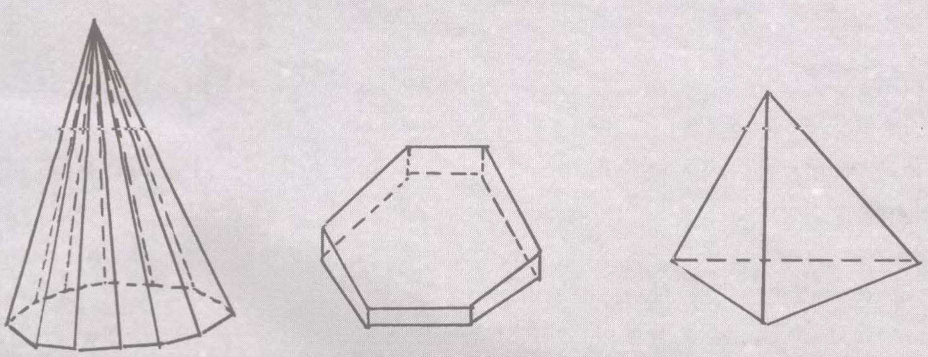
2. Symmetry Has Great Consequences

We shall illustrate that symmetries place great restrictions, and thus severely restrict possibilities.

We consider the transformations of the (three-dimensional Euclidean) space which preserve distance. Obviously translations are such transformations, and since we can move points in the space by translations, we shall only consider those which also fix the origin. The totality of such transformations forms a group, called the orthogonal group in dimension three, denoted by O_3. Some of them may change the orientation of the space. Those which preserve the orientation form a subgroup, called the special orthogonal group in dimension three, denoted by SO_3. It may be identified with the group of rotations of the space.

O_3 also belongs to an infinite family O_n. This is the group of transformations of the n-dimensional Euclidean space which preserve the Euclidean distance and fix the origin at the same time.

Now suppose that we have an object positioned in space with its center of gravity at the origin, then its rotational symmetry group is a subgroup of SO_3. Here are the familiar examples. From a right regular pyramid with an n-sided base we obtain a cyclic group of order n, while a regular plate with n sides exhibits dihedral symmetry and gives D_n. In addition, we have the symmetry groups of the regular solids. The remarkable thing is that these are the only possibilities, provided our object has only a finite amount of symmetry.

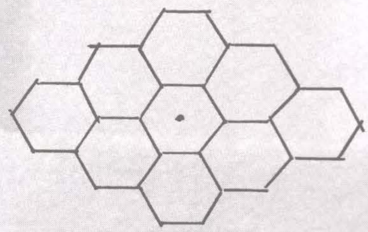


Here is the precise statement.

Theorem: A finite subgroup of SO_3 is either a cyclic group, a dihedral group, or the rotational symmetry group of one of the regular solids.

We offer another example: **wallpaper groups**.

The figure below shows a repeating pattern of hexagons, which if continued indefinitely, fills out the whole plane. The pattern has a certain amount of symmetry, for example, rotation by $2\pi / 6$ (60 degrees) about the origin. This rotation is said to be of order 6, as rotation by 60 degrees 6 times restores the original pattern.



In general we call the symmetry groups of two dimensional repeating patterns wallpaper groups. It turns out that rotations in a wallpaper group are quite special.

Theorem: The order of a rotation in a wallpaper group can only be 2, 3, 4 or 6.

It says in particular that a rotation through the angle $2\pi / 5$ (72 degrees) is not allowed for a wallpaper pattern.

3. Continuous Symmetries - Lie Groups

There are lots of fascinating symmetries in nature, and many of them are continuous. The simplest example is of course O_2, which is also the symmetry group of a circle. The continuous nature of this group is reflected in the fact that the angle of rotation can be arbitrary. Then as we mention in the previous section, there is the orthogonal group in dimension n, which is also the symmetry group of a sphere in the n-dimensional Euclidean space.

Other continuous symmetries are general linear groups and symplectic groups, which like the orthogonal groups, arise from classical geometries. Apart from these, there are five exotic symmetries which do not fit into any pattern.

Mathematically continuous symmetries are represented by Lie groups, and their building blocks are called simple Lie groups. A really remarkable fact about simple Lie groups is that those mentioned above essentially exhaust them. Equally remarkable is the fact that all of them carry very uniform structures, which makes it possible to study continuous symmetries at one go. For this reason, it is easier and much more fruitful to study continuous symmetries than to study finite symmetries.

Now Lie groups enter everywhere in mathematics, notably in geometry, number theory, analysis, and quantum mechanics. To have a bit understanding of why, consider functions on an object with symmetries, be it a geometric object or a quantum mechanical system with a potential. Clearly the most interesting functions on it should reflect the symmetry patterns of the object. Indeed, all special functions of mathematical physics are such functions! For example, Bessel functions are associated to Euclidean motion symmetry, Gegenbauer polynomials to spherical symmetry, Jacobi and Legendre functions to SL(2,R) symmetry, where SL(2,R) is the group of 2x2 real matrices with determinant 1.

Let me end by saying that to exploit continuous symmetry patterns, in particular to understand their consequences, and to solve problems with continuous symmetries built in (either obvious or hidden), is the great task of representation theory of Lie groups.

References

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