COUNTING

by **Koh Khee Meng**

Its Principles & Techniques (9)

23 Climbing Staircase

Let us begin our discussion by considering the following counting problem.

Example 23.1. Figure 23.1 shows a 9-step staircase. A boy wishes to climb the staircase up to the highest step. It is assumed that each move the boy takes can cover only 1 step on 2 steps. How many ways are there for the boy to climb the staircase?

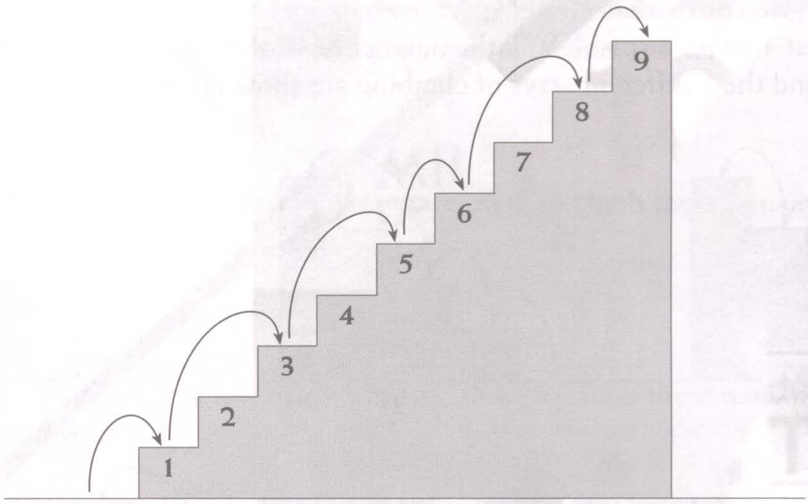
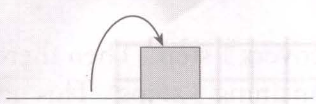


Figure 23.1

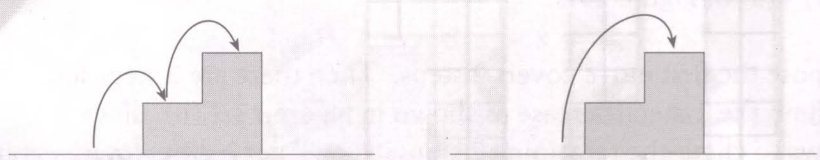
In our previous articles [3-10], we have learnt a number of principles and techniques to solve some counting problems. Naturely, we would like to try and see any of these can be applied to solve the above problem without listing all the possible ways. After poundering for a while, however, we may be doubtful about it. Is there any new idea available to tackle the problem?

The number of steps of the staircase given in the problem, which is '9', may be slightly bigger. Why don't we try by starting with some simpler cases to gain some 'feelings'?

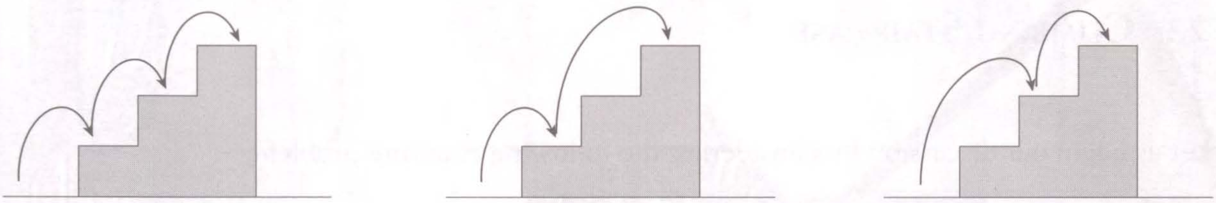
When the staircase consists of 1, 2 and 3 steps, the ways of climbing the staircase are shown in Figure 23.2, and the number of ways is, respectively, 1, 2 and 3 too.



(i) **1-step**



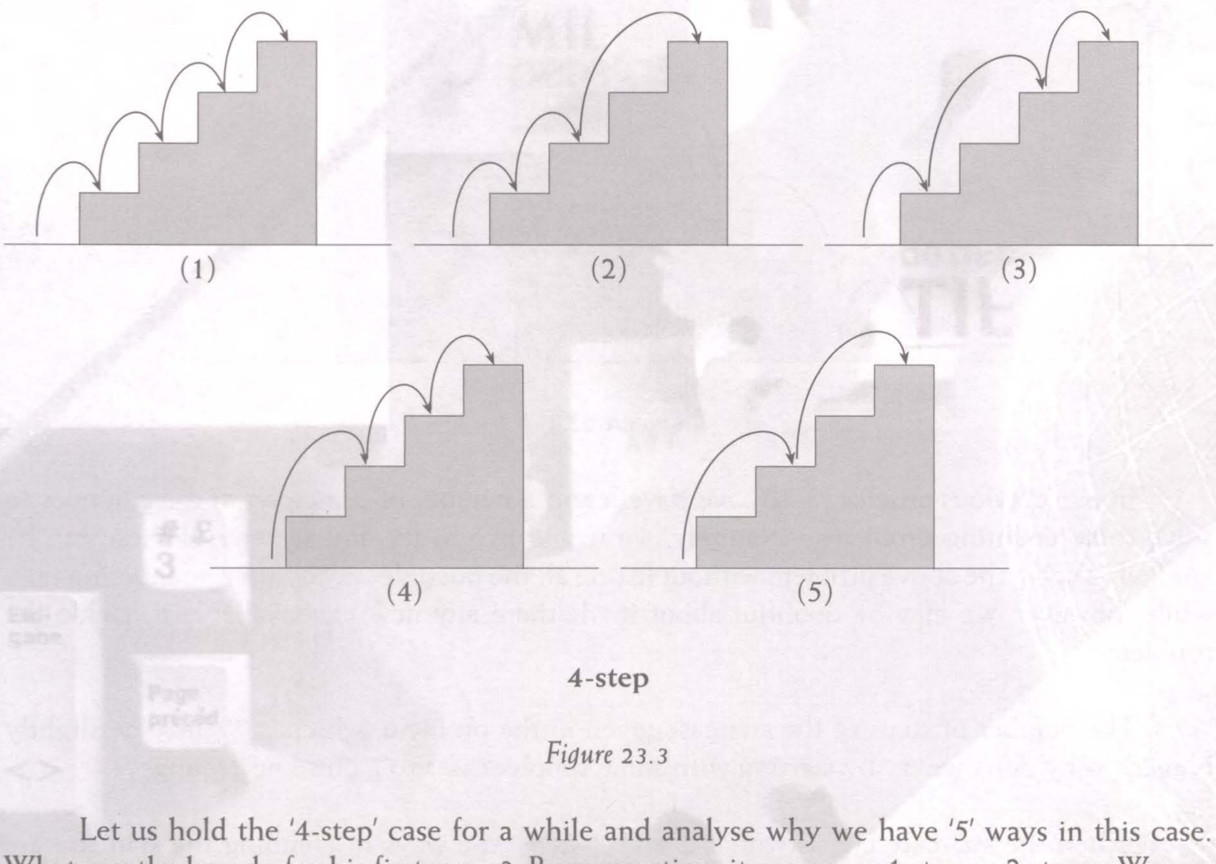
(ii) **2-step**



(iii) **3-step**

Figure 23.2

How about 4-step staircase? Will the number be '4' also? No! The number of ways in this case is now '5' and the 5 different ways of climbing are shown in Figure 23.3.



(1) (2) (3) (4) (5)

4-step

Figure 23.3

Let us hold the '4-step' case for a while and analyse why we have '5' ways in this case. What can the boy do for his first move? By assumption, it can cover 1 step or 2 steps. We now split our consideration into 2 cases accordingly.

(i) Suppose the first move covers 1 step. Then there are 3 steps left. How many ways are there to climb the remaining 3 steps? This question is crucial! Can we link it to the '3-step' case? There are 3 ways to climb the 3-step staircase as shown in Figure 23.2 (iii). If we follow each of these 3 ways to climb the remaining 3 steps, we will get 3 different ways (and no more) to climb the 4-step staircase in this case as shown in (1) - (3) of Figure 23.3.

(ii) Suppose the first move covers 2 steps. Then there are 2 steps left. There are 2 ways to climb the 2-step staircase as shown in Figure 23.2 (ii). If we follow each of these 2 ways to climb the remaining 2 steps, we will get 2 different ways (and no more) to climb the 4-step staircase in this case as shown in (4) - (5) of Figure 23.3. It is now clear that by applying (AP), we will have 3 + 2, i.e., 5 different ways to climb the 4-step staircase.

What have we learnt from the above analysis? We have learnt that the problem of bigger size (4-step) depends on the same problem but of smaller size (3-step and 2-step), and the solution of the problem of bigger size can be obtained from the solutions of the same problem but of smaller size. This is a 'new' idea for us. It works for '4-steps'. Does it work for any 'n-step'?

Now, given any integer $n \geq 4$, for convenience, let us denote by a_n the number of ways to climb an n-step staircase. Thus our previous records show that $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = 5$. Indeed, we have just witnessed that $a_4 = a_3 + a_2$. Can we get a 'similar' equality for a_n?

Imagine now the boy is to climb an n-step staircase. His first move can cover, by assumption, either 1 step or 2 steps. Divide our consideration into two cases as follows:

Case 1. The first move covers 1 step.

Then there are $n-1$ steps left. How many ways are there to climb these remaining $n-1$ steps? By definition, there are a_{n-1} ways.

Case 2. The first move covers 2 steps.

Then there are $n-2$ steps left. How many ways are there to climb these remaining $n-2$ steps? By definition, the answer is a_{n-2}.

Combining the results of these two cases by applying (AP), we conclude that $a_n = a_{n-1} + a_{n-2}$ for $n \geq 4$.

The original problem asks for the determination of a_9. We shall evaluate it from our general result '$a_n = a_{n-1} + a_{n-2}$' together with some 'initial' values (for instance, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and $a_4 = 5$). Applying our general result successively, we have:

$$\begin{aligned} a_5 &= a_4 + a_3 = 5 + 3 = 8, \\ a_6 &= a_5 + a_4 = 8 + 5 = 13, \\ a_7 &= a_6 + a_5 = 13 + 8 = 21, \\ a_8 &= a_7 + a_6 = 21 + 13 = 34, \end{aligned}$$

and finally

$$a_9 = a_8 + a_7 = 34 + 21 = 55,$$

as required.

In the above example, we obtain a sequence of numbers, namely,

$$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8, a_6 = 13, \ldots$$

and in general $a_n = a_{n-1} + a_{n-2}$. The relation $a_n = a_{n-1} + a_{n-2}$ which expresses a_n for a general n, in terms of some preceding numbers in the sequence (in this case, a_{n-1} and a_{n-2}) is called a *recurrence relation*. As we have witnessed just now, deriving a recurrence relation is a way of solving a class of counting problems.

The sequence of numbers: 1, 2, 3, 5, 8, ... as given above is called the sequence of *Fibonacci* numbers, named after the Italian mathematician Leonardo Fibonacci (1170-1240), a great mathematical innovator during the Middle Ages. Fibonacci was born in Pisa. Around 1192, his father was the director of the Pisan trading colony in Algeria. Hoping that his son could become a businessman, the father brought Fibonacci to Algeria to study mathematics with an Arab master. A few years later, sent by the father on business trips, Fibonacci had several occasions to visit places in such as Egypt, Syria, Greece, Sicily etc, where he took opportunities to learn various numerical systems and methods of calculation. Around 1200, after returning to Pisa, Fibonacci started to write a book entitled 'Liber abbaci' (Book of the Abacus), which was completed in 1202. In this book, one finds the following counting problem about rabbits.

Beginning with a pair of baby rabbits, and assuming that each pair gives birth to a new pair each month starting from the second month of its life, how many pairs of rabbits will there be after one year?

Fibonacci (1170 - 1240)

If we write F_n to denote the number of pairs of rabbits at the end of the nth month, then one can see from Figure 23.4 that $F_1 = 1, F_2 = 1, F_3 = 2, F_4 = 3, F_5 = 5$, etc. Indeed, it can be shown (see Problem 23.2) in general that $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$, which is essentially the same as the recurrence relation $a_n = a_{n-1} + a_{n-2}$ that we derived in Example 23.1. Note that in Example 23.1, our initial values are $a_1 = 1$ and $a_2 = 2$ while in Fibonacci's problem, we have $F_1 = F_2 = 1$.

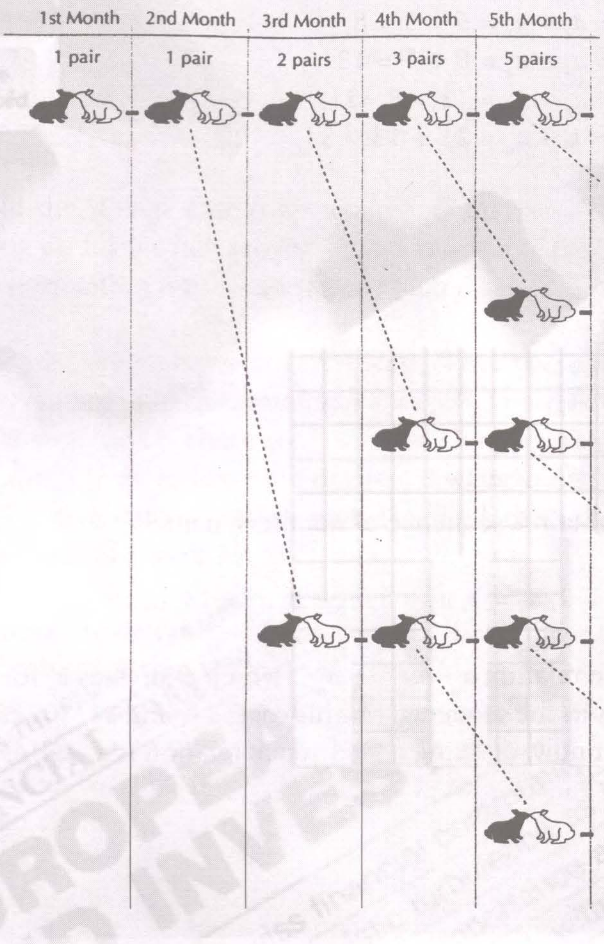


Figure 23.4

Problem 23.1. There are n lines in a plane. Every pair of lines intersect; but no three are meeting at a common point. How many regions is the plane divided into by these n lines?

Problem 23.2. Let F_n denote the number of pairs of rabbits at the end of the nth month, where $n \geq 1$, as given in Fibonacci's problem of rabbits. Show that $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.

Problem 23.3. Find a recurrence relation for the number of binary sequences of length n with no consecutives 0's.

24. THE TOWER OF HANOI

Let us proceed to consider our second example.

Example 24.1. A tower of 8 circular discs of different diameters is stacked on one of the three vertical pegs as shown in Figure 24.1.

Figurer 24.1

The task is to transfer the entire tower to another peg by a number of moves subject to the following rules:

(i) each move carries exactly one disc and
(ii) no disc can be placed on a smaller one.

What is the minimum number of moves required to accomplish the task?

Again, for convenience, let b_n denote the minimum number of moves required to transfer the entire tower with n discs from one peg to another. The problem is to find the value of b_8.

From the experience we have gained in the preceding section, let us first consider some simplest cases. When $n = 1$, it is clear that one move is enough and so $b_1 = 1$ When $n = 2$, one can try and find out that two moves are not enough; whereas the following sequence of moves, as shown in Figure 24.2, shows that three moves could do the job. Thus $b_2 = 3$.

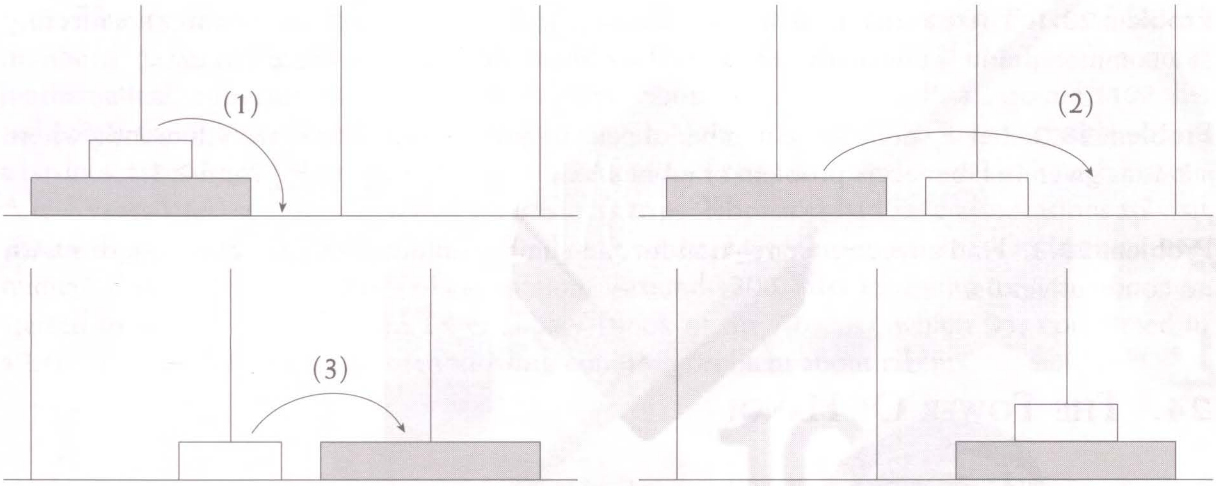


Figure 24.2

Consider now the case when $n = 3$. The sequence of moves shown in Figure 24.3 shows that seven moves are enough to accomplish the task.

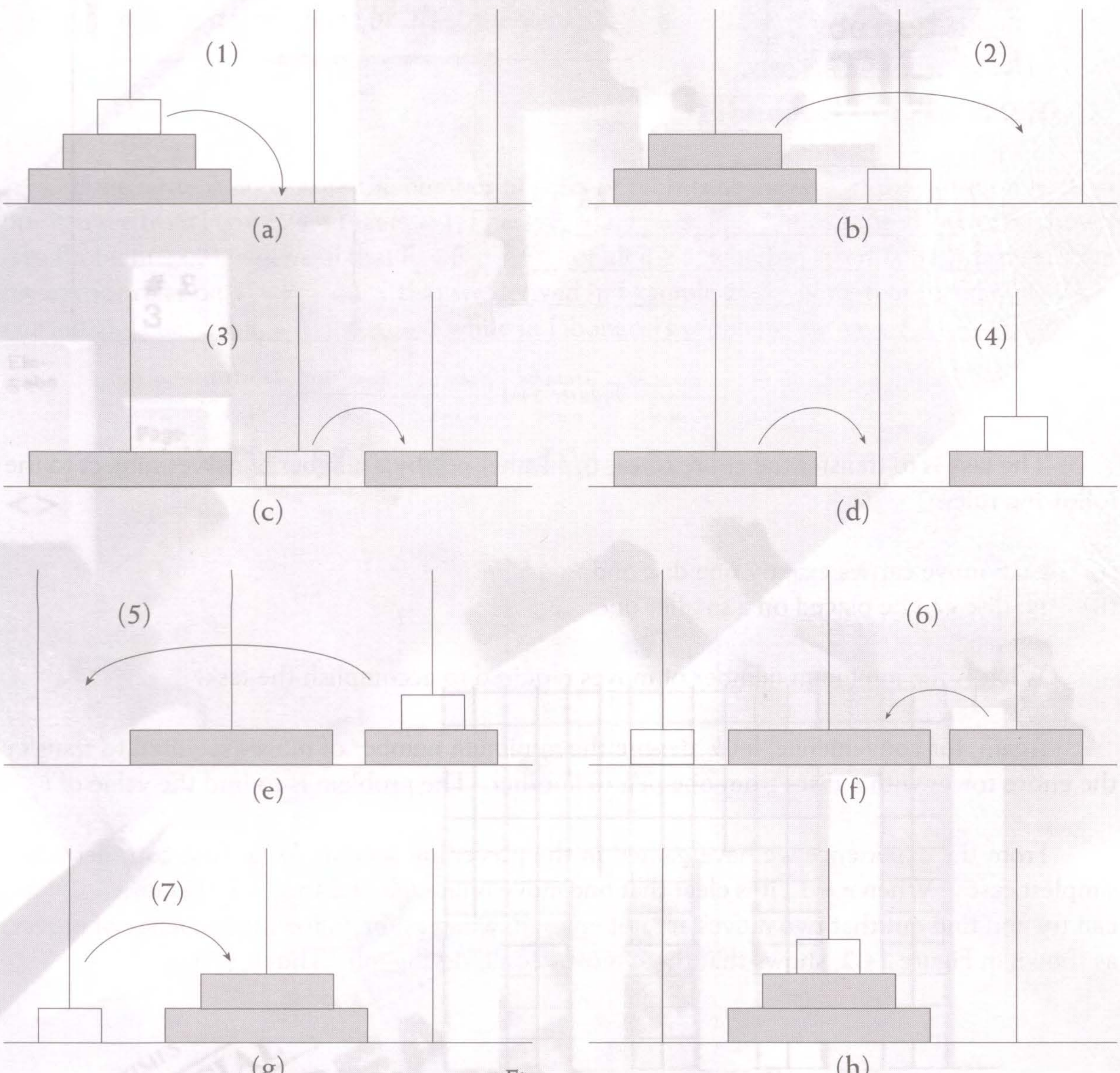


Figure 24.3

Is 'seven' the minimum number of moves required? As shown in Figure 24.3 (d), before the largest disc could be moved to another peg, we have to transfer the entire tower of 2 smaller discs to a peg, and we know that it requires $b_2(=3)$ moves. After moving the largest disc to place at the bottom of another peg as shown in Figure 24.3 (e), we have to transfer the entire tower of 2 smaller discs and place it on the largest disc, and this requires another $b_2(=3)$ moves. Thus we need at least $b_2 + 1 + b_2$ i.e., $2b_2 + 1(=7)$ moves to accomplish the task. This, together with the sequence of seven moves shown in Figure 24.3, shows that $b_3 = 7$.

In the above discussion, we have found that $b_3 = 7$. Indeed, we have obtained the relation $b_3 = 2b_2 + 1$, an instance of a recurrence relation. Can we generalize it? More precisely, given $n \geq 3$, is it true that $b_n = 2b_{n-1} + 1$?

Imagine now we have a tower of n (≥ 3) discs stacked on one of the 3 pegs (say peg (a) as shown in Figure 24.4 and we wish to evaluate b_n, the minimum number of moves needed to transfer the entire tower of n discs to another peg.

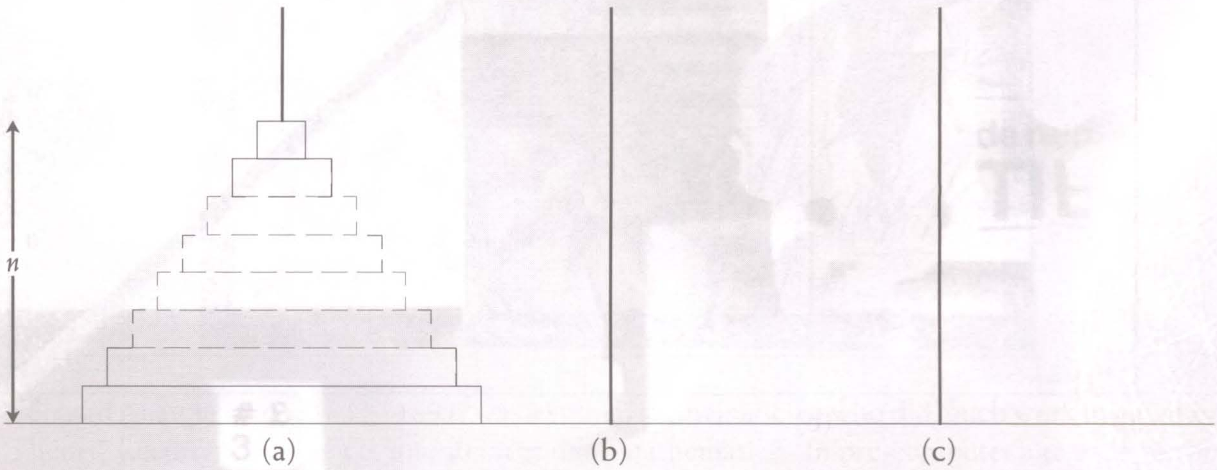


Figure 24.4

In the process of transferring the entire tower, it is clear (by the rule (ii)) that at certain stage we must arrive at the situation, as shown in Figure 24.5, where the entire tower of $n-1$ smaller discs has been transferred to another peg (say peg (c)) so that we have an opportunity to move the largest disc from the original peg to the bottom of a peg (in this case, peg (b)). What is the minimum number of moves needed to transfer the entire tower of $n-1$ smaller discs from peg (a) to peg (c)? By definition, this number is b_{n-1}. After moving the largest disc from peg (a) to peg (b) as shown in Figure 24.6, our final job is to transfer the entire tower of discs at peg (c) and place it on the largest disc at peg (b). What is the minimum number of moves needed for this final job? By definition, this number is b_{n-1} again. Summing up, we see that the minimum number of moves that are needed for the whole task is $b_{n-1} + 1 + b_{n-1}$. Accordingly, by definition of b_n, we have

$$b_n = 2b_{n-1} + 1,$$

another example of a recurrence relation. Let us return to the problem in Example 24.1, where we were asked to evaluate b_8. Based on the result that $b_2 = 3$, by applying our recurrence relation $b_n = 2b_{n-1} + 1$ successively, we obtain

$$b_2 = 3, b_3 = 7, b_4 = 15, b_5 = 31, b_6 = 63, b_7 = 127,$$

and finally $b_8 = 255$, as required.

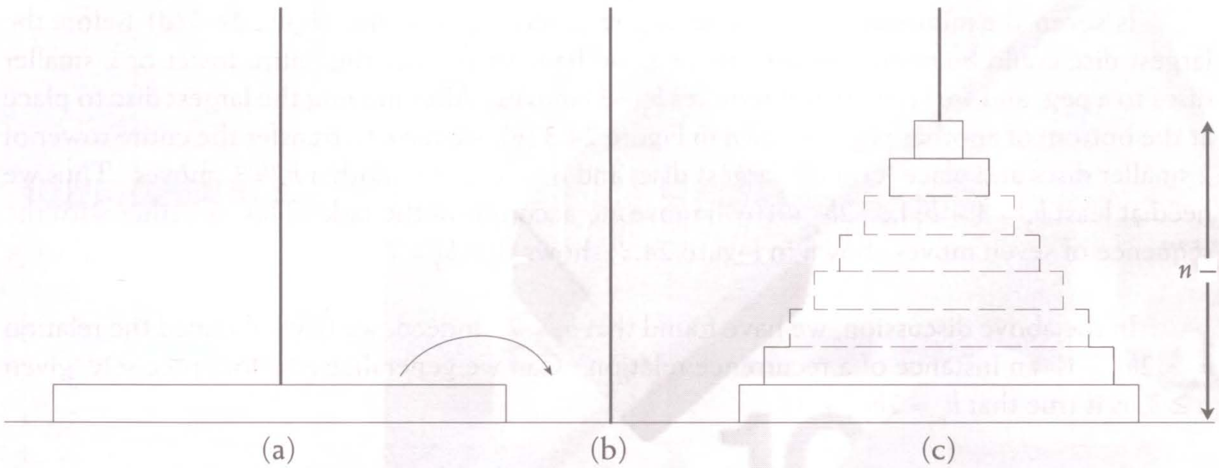


Figure 24.5

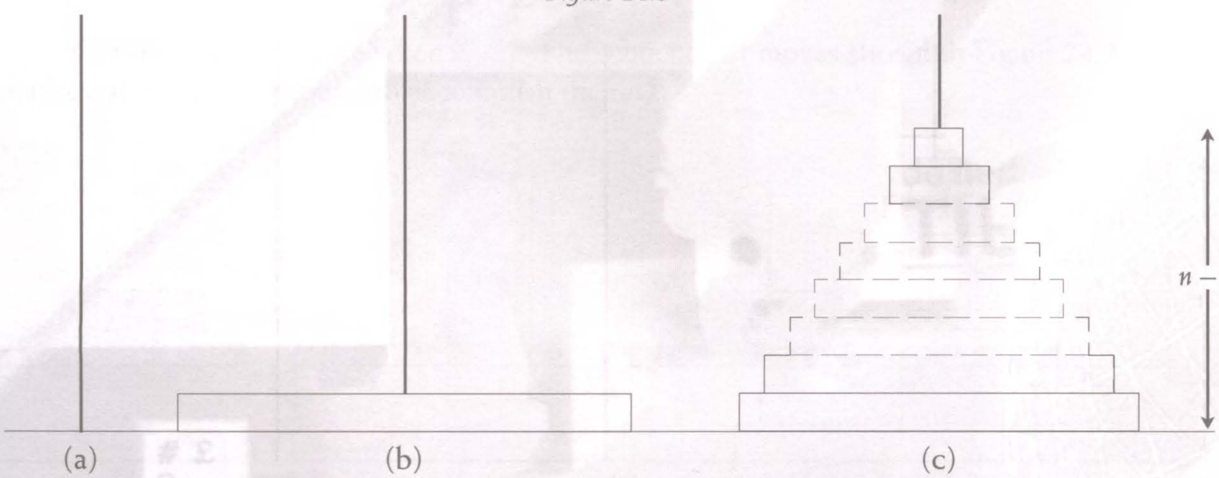


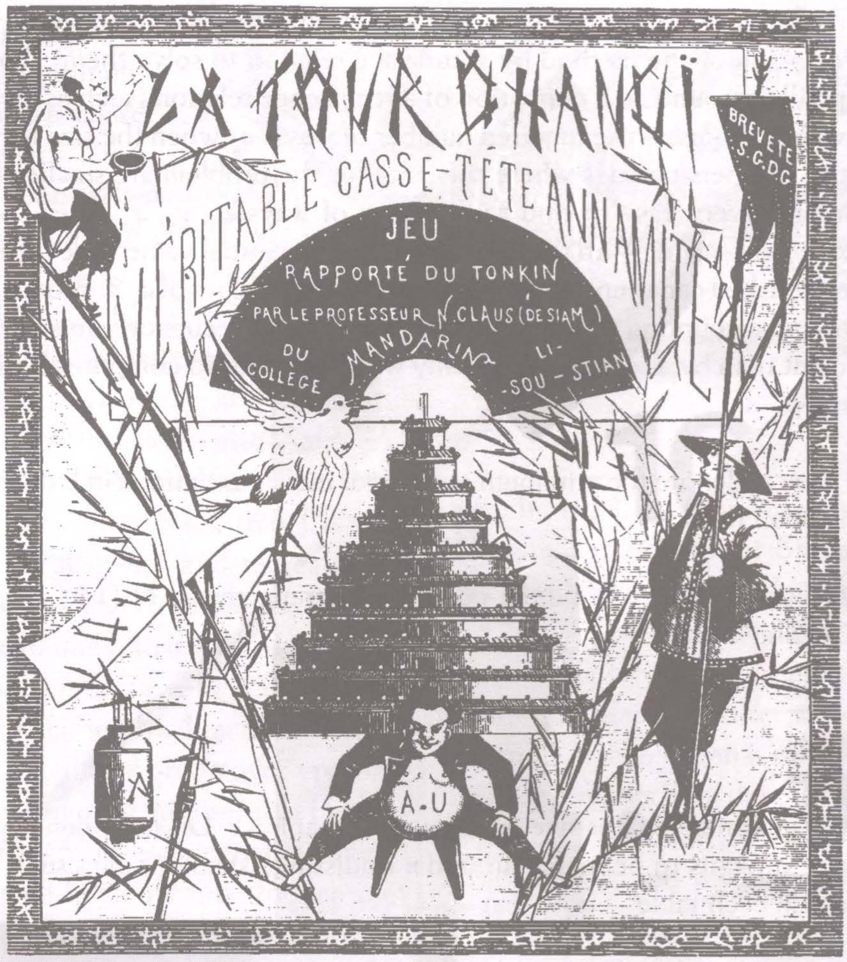
Figure 24.6

Observe that we actually have a nice single formula for the value of b_n :

$$b_2 = 3 = 2^2 - 1, b_3 = 7 = 2^3 - 1, b_4 = 15 = 2^4 - 1, \dots ;$$

and in general, $b_n = 2^n - 1$ (see Problem 24.1).

The problem described in Example 24.1 is known as the Tower of Hanoi (ToH). Why is 'Hanoi', the capital of Vietnam, associated to this problem? Well, this could have something to do with the following two facts that the inventor of this problem is a French and the problem was introduced at the time when France began her military involvement in Vietnam (see Hinz [1]). According to Hinz [2], the picture shown in Figure 24.7, which was the cover of a box, was found in Paris in 1883. Looking at the picture closely, we could find several items therein which are related to tropical Asia, and particularly, Vietnam. These include a Vietnamese, two sites in Vietnam: Tonkin and Annam, and the title 'La Tour d'Hanoi'. Two special names are also appeared in the picture. These are Professor N. Claus (de Siam) and his College Li-Sou-Stian. According to the French mathematician de Parville, the two names above are anagrams for Professor Lucas (d'Amiens), the inventor of this problem, and his College Saint Louis. As Lucas was Agrégé de l'Université, it is believed that he is the one carrying the ten-level tower in the picture. Francois



Édouard Anatole Lucas (1842 - 1891) was a French mathematician who did much work in Number Theory, Recurrent Sequences and Recreational Mathematics. In pre-computer age, he was the last 'largest prime number record' holder. He gave the sequence of Fibonacci numbers, as introduced before, its name and he himself has the following sequence: 2, 1, 3, 4, 7, 11, 18, 29, ... named after him.

Édouard Lucas (1842 - 1891)

In this article, we have discussed two counting problems and introduced a way, called the technique of recursion or the method by recurrence relation to solve them. The technique of recursion essentially amounts to a derivation of a recurrence relation (such as $a_n = a_{n-1} + a_{n-2}$ and $a_n = 2a_{n-1} + 1$) which expresses the required number of ways 'a_n' when the size for the problem is n in terms of the numbers of ways where the sizes for the problem are smaller than n (such as a_{n-1}, a_{n-2}). It is often very easy to find the number of ways 'a_1, a_2, a_3' when the sizes for the problem are extremely small. With these values, called the initial values, the recurrence relation that has been established can then generate successively the values of a_n's. From a computational stand point, solving a counting problem by the technique of recursion can sometimes be more useful and efficient than by a formula, especially when we need to compute all the values: a_1, a_2, ..., a_n up to some point.

Problem 24.1. Let b_n denote the minimum number of moves as defined in Example 24.1. Show that $b_n = 2^n - 1$ for all $n \geq 1$.

Problem 24.2. For $n \geq 3$, let D_n be the number of derangements of $\{1,2,\ldots,n\}$, as defined in Section 22 [10]. Show that

$$D_n = (n-1)(D_{n-1} + D_{n-2}).$$

Problem 24.3. Let r and n be positive integers such that $n \leq r$. Let $s(r,n)$ denote the number of ways to arrange r students to be seated around n (indistinguishable) tables such that there is at least one student in each table. Show that

$$s(r,n) = s(r-1, n-1) + (r-1)s(r-1, n).$$

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