Competition Corner

by
Tay Tiong Seng

In this issue we publish the problems of the 11th Irish Mathematical Olympiad, March 1998 and the XI Asia Pacific Mathematical Olympiad, March 1999. We also present solutions of the Auckland Mathematical Olympiad 1998, Ukrainian Mathematical Olympiad 1997 as well as the 39th International Mathematical Olympiad. These were published in the last issue. Please send your solutions of the Irish Mathematical Olympiad to me at the address given below. All correct solutions will be acknowledged.

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Problems

Eleventh Irish Mathematical Olympiad, May 1998

First Paper

1. Show that if x is a nonzero real number, then

$$x^8 - x^5 - \frac{1}{x} + \frac{1}{x^4} \geq 0.$$

2. P is a point inside an equilateral triangle such that the distances from P to the three vertices are 3, 4 and 5, respectively. Find the area of the triangle.

3. Show that no integer of the form $xyxy$ in base 10 (where x and y are digits) can be the cube of an integer.

Find the smallest base $b > 1$ for which there is a perfect cube of the form $xyxy$ in base b.

4. Show that a disc of radius 2 can be covered by seven (possibly overlapping) discs of radius 1.

5. If x is a real number such that $x^2 - x$ is an integer, and, for some $n \geq 3$, $x^n - x$ is also an integer, prove that x is an integer.

Second paper

6. Find all positive integers n that have exactly 16 positive integral divisors $d_1, d_2, \ldots, d_{16}$ such that

$$1 = d_1 < d_2 < \cdots < d_{16} = n,$$

$d_6 = 18$, $d_9 - d_8 = 17$.

7. Prove that if a, b, c are positive real numbers, then

$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right) \tag{1}$$

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \leq \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \tag{2}$$

8. Let N be the set of natural numbers (i.e., the positive integers).

(a) Prove that N can be written as a union of three mutually disjoint sets such that, if $m, n \in \mathbb{N}$ and $|m - n| = 2$ or 5, then m and n are in different sets.

(b) Prove that N can be written as a union of four mutually disjoint sets such that, if $m, n \in \mathbb{N}$ and $|m - n| = 2, 3$, or 5, then m and n are in different sets. Show however, that it is impossible to write N as a union of three mutually disjoint sets with this property.

9. A sequence of real numbers x_n is defined recursively as follows: x_0 and x_1 are arbitrary positive real numbers, and

$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}, \quad n = 0, 1, 2, \ldots.$$

Find x_{1998}.

10. A triangle ABC has positive integer sides, $\angle A = 2\angle B$ and $\angle C > 90°$. Find the minimum length of the perimeter.

XI Asian Pacific Mathematical Olympiad, March 1999

1. Find the smallest positive integer n with the following property: There does not exist an arithmetic progression of 1999 terms of real numbers containing exactly n integers.

2. Let $a_1, a_2, \ldots$ be a sequence of real numbers satisfying $a_{i+j} \leq a_i + a_j$ for all $j = 1, 2, \ldots$. Prove that

$$a_1 + \frac{a_2}{2} + \frac{a_3}{3} + \cdots + \frac{a_n}{n} \geq a_n$$

for each positive integer n.

3. Let Γ_1 and Γ_2 be two circles intersecting at P and Q. The common tangent closer to P, of Γ_1 and Γ_2 touches Γ_1 at A and Γ_2 at B. The tangent of Γ_1 at P meets Γ_2 at C, which is different from P and the extension of AP meets BC at R. Prove that the circumcircle of triangle PQR is tangent to BP and BR.

4. Determine all pairs (a, b) of integers with the property that the numbers $a^2 + 4b$ and $b^2 + 4a$ are both perfect squares.

5. Let S be a set of $2n + 1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of S on its circumference, $n - 1$ points in its interior and $n - 1$ in its exterior. Prove that the number of good circles has the same parity as n.

Solutions

Auckland Mathematical Olympiad 1998, Division 2

6. Find all real solutions of the system of equations

$$x + y + xy = 11 \quad (1)$$
$$x^2 + xy + y^2 = 19 \quad (2)$$

Solution similar to the official solution by Lim Chong Jie (Temasek Junior College), Lim Kim Huat (National Junior College), the following students from Raffles Institution: Ong Chin Siang, Colin Tan Weiyu, Justin Yek Jia Jin; and the following students from Anglo Chinese School (Independent): Joel Tay Wei En, Julius Poh Wei Quan. Chan Sing Chun gave a different solution.

Adding up the two equations we get

$$(x+y)^2 + (x+y) - 30 = (x+y+6)(x+y-5) = 0.$$

Thus $x+y=-6$ or 5. If $x+y=-6$, then (1) yields

$$x^2 + 6x + 17 = 0$$

which has no real roots. If $x+y=5$, then (1) yields

$$x^2 - 5x + 6 = 0.$$

The solutions are $x = 2, 3$. Thus $(x,y) = (2,3)$ or $(3,2)$ both of which satisfy (2). Thus these are the only solutions.

7. Some cells of an infinite square grid are coloured black and the rest are coloured white so that each rectangle consisting of 6 cells (2×3 or 3×2) contains exactly 2 black cells. How many black cells might a 9×11 rectangle contain?

Solution similar to the official one by: He Ruijie (Dunman High School), Lim Chong Jie (Temasek Junior College), Julius Poh Wei Quan and Joel Tay Wei En from Anglo Chinese School (Independent) and the following students from Raffles Institution: Ong Chin Siang, Colin Tan Weiyu, Justin Yek Jia Jin.

Suppose there is a pair of adjacent black squares. (See Figure 1) Then the remaining 7 cells in the 3×3 rectangle are white, which is impossible since the 3×2 rectangle formed by the last two columns contains only one black cell.

Suppose there is a white cell sandwiched between a pair of black cells. Then we have the situation shown in Figure 2 which is again impossible. Suppose there is a row of three white cells (first row of Figure 3). Then there must be two black cells in the second row which is impossible (by the previous two cases). Thus any 1×3 rectangle can contain only 1 black cell. Since in a 9×11 rectangle, there are 33 1×3 rectangle, it can contain 33 black cells. Figure 4 shows a colouring pattern which gives 33 black cells.

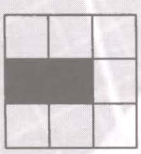


Figure 1

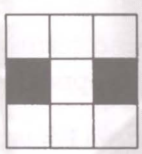


Figure 2

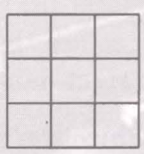


Figure 3

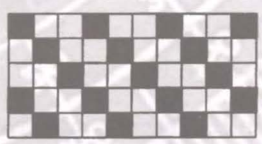
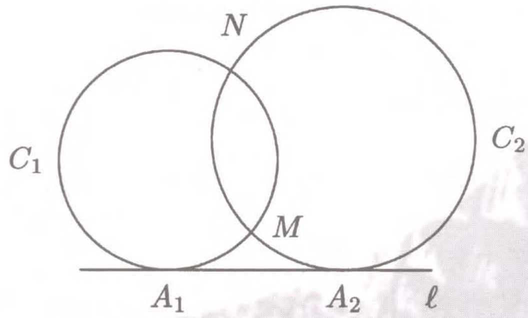


Figure 4

8. Two circles C_1 and C_2 of radii r_1 and r_2 touch a line ℓ at points A_1 and A_2, as shown in the figure below.



The circles intersect at points M, N. Prove that the circumradius of the triangle A_1MA_2 does not depend on the length of A_1A_2 and is equal to $\sqrt{r_1r_2}$.

Solutions similar to the official one by Chan Sing Chun, A. Robert Pargeter (England) and the following students from Raffles Institution: Colin Tan Weiyu, Justin Yek Jia Jin.

Let $\angle MA_1A_2 = \theta$ and $\angle MA_2A_1 = \phi$. If O is the centre of C_1, then $\angle A_1OM = 2\theta$. Thus $MA_1 = 2r_1 \sin\theta$. Similarly $MA_2 = 2r_2 \sin\phi$. Let r be the circumradius of ΔMA_1A_2. Then, by sine rule, we have

$$\frac{MA_1}{\sin\phi} = \frac{MA_2}{\sin\theta} = 2r.$$

A simple calculation then yields $r = \sqrt{r_1r_2}$.

9. Let α and β be two acute angles such that $\sin^2\alpha + \sin^2\beta = \sin(\alpha+\beta)$. Prove that $\alpha + \beta = \pi/2$.

Solution by Lim Chong Jie (Temasek Junior College). Also solved by Lim Kim Huat (national Junior College). Ong Chin Siang and Colin Tan Weiyu both of Raffles Institution obtained solution similar to the official one.

Since $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \sin\beta\cos\alpha$, the equation becomes:

$$\sin\alpha(\sin\alpha - \cos\beta) = \sin\beta(\cos\alpha - \sin\beta),$$

i.e.,

$$2\sin\alpha\sin\frac{2\alpha+2\beta-\pi}{4}\cos\frac{2\alpha-2\beta+\pi}{4} = -2\sin\beta\sin\frac{2\alpha+2\beta-\pi}{4}\cos\frac{2\beta-2\alpha+\pi}{4}.$$

Since $\sin\alpha$, $\sin\beta$ are both positive, as are the two cosine terms, we have $\sin\frac{2\alpha+2\beta-\pi}{4} = 0$ otherwise the left hand side and the right hand side have opposite signs. Thus $\alpha+\beta = \pi/2$.

10. Find all prime numbers p for which the number p^2+11 has exactly 6 different divisors (including 1 and the number itself).

Solution similar to the official solution by Chan Sing Chun, Joel Tay Wei En (Anglo Chinese School (Independent)) and the following students from Raffles Institution: Ong Chin Siang, Colin Tan Wei Yu, Justin Yek Jia Jin.

If $p = 3$, $p^2 + 11$ has exactly 6 divisors. Now let $p > 3$. Then p is odd and $p^2 \equiv 1 \pmod 4$, thus $p^2 + 11 \equiv 0 \pmod 4$. Also $p^2 \equiv 1 \pmod 3$, thus $p^2 + 11 \equiv 0 \pmod 3$. Thus $p^2 + 11 \equiv 0 \pmod{12}$. Since every $p^2 + 11 > 12$ and every divisor of 12 is also a divisor of $p^2 + 11$, it follows that $p^2 + 11$ has more divisors than 12. Thus $p^2 + 11$ has more than 6 divisors. The only prime number with the desired property is therefore 3.

Ukrainian Mathematical Olympiad, 1997 (Selected problems)

1. (9th grade) Consider a rectangular board in which the cells are coloured black and white alternately like chess board cells. In each cell an integer is written. It is known that the sum of the numbers in every row and every column is even. Prove that sum of all numbers in the black cells is even.

Solution by SIMO problem group. Call a cell odd if the number written in it is odd. A collection of cells is called a *loop* if each column and each row contains either 2 or none of the cells in the collection. Consider only the odd cells. In each column, as well as in each row, the number of odd cells is even. Thus the odd cells can be partitioned into loops. The proof goes as follows. Start with any odd cell. In its column there is another odd cell since the number of odd cells in that column is even. Similarly, in the row containing the second odd cell, there is another odd cell, not previously chosen. Continuing this way, we must eventually return to the starting odd cell. However, there may be a column or row with more than two odd cells. Say a is the first cell and b is the last cell in the column belonging to the collection in the order in which they were selected. Then we simply drop all the cells in the collection after a and before b. In this way, we can arrive at a collection which is a loop. This gives us a loop. Removing this loop leaves us in the same situation as before, i.e., every column and every row contain an even number of odd cells. Thus we can continue to find another loop. We can continue this way until there are no odd cells. Thus the odd cells can be partitioned into loops.

Next we prove that in any loop the number of white cells is even as is the number of black cells. If a column contains two cells of the same colour, then the distance between the two cells is even. If it contains a white and a black cell, the distance between the cells is odd. Since we start and end with the same cell, the net vertical distance traveled is 0. Hence, the number of columns which contain a white and a black cell of a loop must be even. Thus the number of white cells and the number of black cells in a loop are both even.

From the above we conclude that the number of black odd cells is even. Thus the sum of the numbers in the black cells is even.

Official solution. Since the official solution is very elegant, we also present the official solution. Let the $(1, 1)$ cell, i.e., the cell at the top left corner, be black. Add up the odd columns and the even rows. Then each black cell appears exactly once in the sum and each white cell appears either twice or none at all in the sum. Since the sum is even, the sum of the numbers in the black cells is even. (Since the sum of all rows is even, one can see that the sum of all white cells is also even. Thus the same conclusion can be shown if the $(1,1)$ cell is white.)

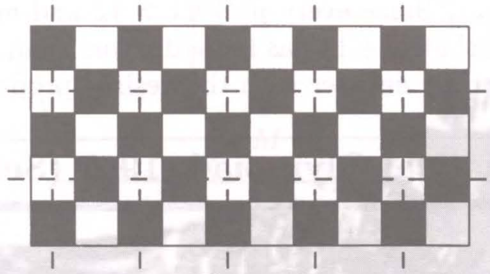


Figure 4

2. (10th grade) Solve the system of equations in real numbers:

$$x_1 + x_2 + \cdots + x_{1997} = 1997 \tag{1}$$
$$x_1^4 + x_2^4 + \cdots + x_{1997}^4 = x_1^3 + x_2^3 + \cdots + x_{1997}^3 \tag{2}$$

Solution similar to the official one by: Colin Tan Weiyu, Justin Yek Jia Jin both of Raffles Institution.

From (1) and (2) we have

$$(x_1 - 1) + (x_2 - 1) + \cdots + (x_{1997} - 1) = 0 \tag{3}$$
$$x_1^3(x_1 - 1) + \cdots + x_{1997}^3(x_{1997} - 1) = 0. \tag{4}$$

From (3) and (4), we have

$$(x_1 - 1)^2(x_1^2 + x_1 + 1) + \cdots + (x_{1997} - 1)^2(x_{1997}^2 + x_{1997} + 1) = 0. \tag{5}$$

Since $x^2 + x + 1 > 0$ and $(x-1)^2 \geq 0$, for (5) to hold, we must have $x_i - 1 = 0$ for $i = 1, 2, \ldots, 1997$. Indeed these also satisfy the original equations (1) and (2).

3. (10th grade) Let $d(n)$ denote the greatest odd divisor of the natural number n. Define the function $f : \mathbb{N} \to \mathbb{N}$ by $f(2n-1) = 2^n$, $f(2n) = n + 2n/d(n)$ for all $n \in \mathbb{N}$. Find all k such that $f(f(\ldots(1)\ldots)) = 1997$ where f is iterated k times.

Answer obtained with proof by Joel Tay Wei En (Anglo Chinese School (Independent)), Justin Yek Jia Jin and Colin Tan Weiyu both from Raffles Institution.

Let $a_1 = 1$ and $a_{n+1} = f(a_n)$ for $n \geq 1$. Then $a_1 = 1$, $a_2 = 2$, $a_3 = 3$ and so on. After a few more terms, it is easy to notice that if $a_m = 2^j$, then $a_{m+j+1} = 2^{j+1}$. This can be proved as follows. Let $a_m = 2^j$. Then

$$a_{m+1} = 2^{j-1} + 2^j = 3 \cdot 2^{j-1};$$
$$a_{m+2} = 3 \cdot 2^{j-2} + 2^{j-1} = 5 \cdot 2^{j-2};$$

Thus we can formulate the following induction hypothesis:

$$a_{m+i} = (2i+1)2^{j-i}, \quad i = 0, 1, \ldots, j.$$

$$a_{m+i+1} = (2i+1)2^{j-i-1} + 2^{j-i} = (2(i+1)+1)2^{j-i-1}.$$

Thus the result follows by induction. Since $a_1 = 2^0$, we can write down a formula for a_n as follows. If $n = (0+1+2+\cdots+p)+1+q$, where $p \geq 0$, $0 \leq q \leq p$, then $a_n = (2q+1)2^{p-q}$. If $(2q+1)2^{p-q} = 1997$, then $p = q = 998$. Thus $a_{499500} = 1997$. Thus $k = 499499$.

6. (11th grade) Let $\mathbb{Q}^+$ denote the set of all positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$, $f(x+1) = f(x)+1$, and $f(x^2) = (f(x))^2$.

Solution by Ng Boon Leong (Anglo Chinese School (Independent)).

First note that from $f(x+1) = f(x)+1$, we can prove by induction that $f(x+m) = f(x)+m$. Now consider a positive rational number $r = a/b$ where a and b are positive integers without common divisors other than 1. (Note that when $b = 1$, r is a positive integer.) Let $f(r) = x$. Then

$$f((b+r)^2) = f(b^2+2a+r^2) = b^2+2a+f(r^2) = b^2+2a+(f(r))^2 = b^2+2a+x^2$$

and

$$(f(b+r))^2 = (b+f(r))^2 = (b+x)^2 = b^2+2bx+x^2.$$

Comparing the two, we have $x = a/b$. Thus $f(r) = r$. Clearly this function satisfy the conditions given in the problem.

7. (11th grade) Find the minimum value of n such that in any set of n integers there exist 18 integers with sum divisible by 18.

Solution Thirty four numbers are insufficient as shown by a collection of seventeen 0's and seventeen 1's. Now we shall show that 35 numbers are sufficient. Let $a_1, \ldots, a_{35}$ be any given set of 35 integers. Let a_{36} be a number chosen so that the sum $\sum_{i=1}^{36} a_i$ is divisible by 36. Among any 5 integers, there are always three whose sum is divisible by 3. (This fact can be proved by considering the remainders of the numbers when divided by 3. If there are three remainders with the same value, then the corresponding three numbers sum to a multiple of three. If not, then there are three numbers with pairwise distinct remainders. Again they sum to a multiple of 3.) Thus we can divide the 36 integers into twelve groups $A_1, \ldots, A_{12}$, each with three numbers whose sum is divisible by 3. By a similar argument, we can divide $A_1, \ldots, A_{12}$, into four groups $B_1, \ldots, B_4$, each consisting of three A_i's such that the sum of the nine numbers is divisible by 9. Consider the sum of the numbers in each of the B_i's. Two of them must be of same parity, as are the other two. Thus the 36 numbers can be divided into two groups, each with 18 numbers whose sum is divisible by 18. The group that does not contain a_{36} contains 18 numbers whose sum is divisible by 18.

39th International Mathematical Olympiad, 1998

1. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

First solution: If $ABCD$ is a cyclic quad, then it is easy to show that $\angle APB + \angle CPD = 180°$. From here one easily concludes that the two areas are equal.

For the converse we use coordinate geometry. Let P be the origin. Let the coordinates of A and B be $(-a,-b)$ and $(a,-b)$, respectively where a and b are both positive. Let the midpoint of CD be (c,d). Then, since P is in the interior, C is $(c,d) - t(-d,c) = (c+td, d-tc)$ and D is $(c,d) + t(-d,c) = (c-td, d+tc)$, where $t > 0$. (The vector CD is in the direction $(-d,c)$.) Without loss of generality, let $c^2 + d^2 = 1$. Then area of PCD and APB are t and ab, respectively. Thus $t = ab$. The fact that AC is perpendicular to BD implies that

$$(c - td - a, d + tc + b) \cdot (c + td + a, d - tc + b) = 0.$$

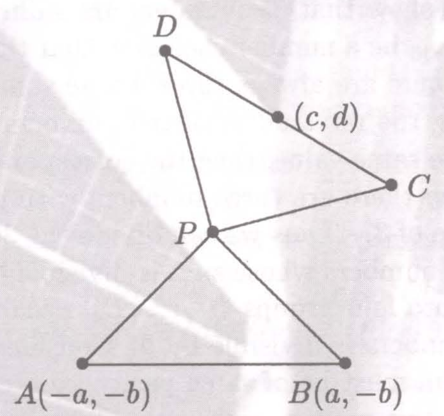
This simplifies to

$$(1 - a^2)(b^2 + 2bd + 1) = 0.$$

We have

$$PA = PB = a^2 + b^2, \quad PC = PD = t^2 + 1 = a^2b^2 + 1.$$

Thus $PA = PB = PC = PD = b^2 + 1$ when $a^2 = 1$, i.e., A, B, C, D are on a circle with centre at P.



We now consider the case $b^2 + 2bd + 1 = 0$. Consider this as a quadratic equation in b, the discriminant $4d^2 - 4 \geq 0$ if and only if $d^2 \geq 1$. But we know that $d^2 \leq 1$. Thus $d^2 = 1$ and consequently $b = \pm 1$ or $b^2 = 1$. Since $b > 0$, we actually have $b = 1$ and $d = -1$. Thus $c = 0$ whence $A = C$ and $B = D$, which is impossible.

Second solution (official): Let AC and BD meet at E. Assume by symmetry that P lies in ΔBEC and denote $\angle ABE = \phi$ and $\angle ACD = \psi$. The triangles ABP and CDP are isosceles. If M and N are the respective midpoints of their bases AB and CD, then $PM \perp AB$ and $PN \perp CD$. Note that M, N and P are not collinear due to the uniqueness of P.

Consider the median EM to the hypotenuse of the right triangle ABE. We have $\angle BEM = \phi$, $\angle AME = 2\phi$ and $\angle EMP = 90° - 2\phi$. Likewise, $\angle CEN = \psi$, $\angle DNE = \psi$ and $\angle ENP = 90° - 2\psi$. Hence $\angle MEN = 90° + \phi + \psi$ and a direct computation yields

$$\angle NPM = 360° - (\angle EMP + \angle MEN + \angle ENP) = 90° + \phi + \psi = \angle MEN.$$

It turns out that, whenever $AC \perp BD$, the quadrilateral $EMPN$ has a pair of equal opposite angles, the ones at E and P.

We now prove our claim. Since $AB = 2EM$ and $CD = 2EN$, we have $[ABP] = [CDP]$ if and only if $EM \cdot PM = EN \cdot PN$, or $EM/EN = PN/PM$. On account of $\angle MEN = \angle NPM$, the latter is equivalent to $\Delta EMN \sim \Delta PNM$. This holds if and only if $\angle EMN = \angle PNM$ and $\angle ENM = \angle PMN$, and these in turn mean that $EMPN$ is a parallelogram. But the opposite angles of $EMPN$ at E and P are always equal, as noted above. So it is a parallelogram if and only if $\angle EMP = \angle ENP$; that is, if $90° - 2\phi = 90° - 2\psi$. We thus obtain a condition equivalent to $\phi = \psi$, or to $ABCD$ being cyclic.

2. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

Solution: Form a matrix where columns represent the contestants and the rows represents the judges. And we have a 1 when the judge "passes" the corresponding contestant and a 0 otherwise. A pair of entries in the same column are "good" if they are equal. Thus the number of good pairs in any two rows is at most k whence the total number of good pairs in the matrix is at most $\binom{b}{2}k = kb(b-1)/2$. In any column, if there are i zeroes, then the total number of good pairs is $\binom{i}{2} + \binom{j}{2}$, where $j = b - i$. Write $b = 2m + 1$ (since b is odd), we have

$$\binom{i}{2} + \binom{j}{2} - m^2 = (m-i)^2 + (m-i) = (m-j)^2 + (m-j) \geq 0$$

since either $m - i \geq 0$ or $m - j \geq 0$. Thus the total number of good pairs is at least $am^2 = a(b-1)^2/4$. Therefore

$$a(b-1)^2/4 \leq kb(b-1)/2$$

from which the result follows.

3. For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that

$$\frac{d(n^2)}{d(n)} = k$$

for some n.

Solution: Note that if $n = p_1^{k_1} \cdots p_i^{k_i}$ is the prime decomposition of n, then $d(n) = (k_1 + 1) \cdots (k_i + 1)$ and $d(n^2) = (2k_1 + 1) \cdots (2k_i + 1)$ Thus for $d(n^2)/d(n)$ to be an integer, $k_1, \ldots, k_i$ must all be even so that the denominator contains no even divisors. Thus, an integer q satisfies $d(n^2)/d(n) = q$ for some n if and only if q is of the form

$$\frac{(4k_1+1)(4k_2+1)\ldots(4k_i+1)}{(2k_1+1)(2k_2+1)\ldots(2k_i+1)} \qquad (*)$$

Thus q is necessarily odd. Hence we need to show that every odd number can be expressed in the same way. Certainly 1 and 3 can be so expressed as $1 = 1/1$ and $3 = \frac{5}{3}\frac{9}{5}$. Let p be an odd integer. We assume that every odd integer less than p can be written in the form $(*)$. We have

$$p + 1 = 2^m(2k+1)$$

for some positive integer m and nonnegative integer k. If $m = 1$, then $p = 4k+1 = \frac{4k+1}{2k+1}(2k+1)$. Since $2k+1 < p$, by the induction hypothesis, it can be expresses in the form $(*)$ and hence so can p.

Now suppose that $m > 1$. We have

$$p(2^m - 1) = 2^{2m-1}k - 2^m k + 2^{2m-2} - 2^m + 1 = 2^m x + 1$$

and

$$\frac{2^m x + 1}{2^{m-1}x+1}\,\frac{2^{m-1}x+1}{2^{m-2}x+1}\cdots\frac{4x+1}{2x+1} = \frac{p(2^m-1)}{2x+1} = \frac{p}{2k+1}$$

since $2x + 1 = (2^{m-1} - 1)(2k+1)$. Since the left hand side is of the form $(*)$ and $2k+1$ can be written in that form by the induction hypothesis, we conclude that p can also be written in the same form.

(Note: The main idea is that it is easy to solve the case where $p \equiv 1 \pmod 4$. For $p \equiv 3 \pmod 4$, we try to multiply p with an odd integer so that $p(4k+3) = 4\ell + 1$. By considering small values of p it was found that $2^m - 1$ as defined above works.)

4. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

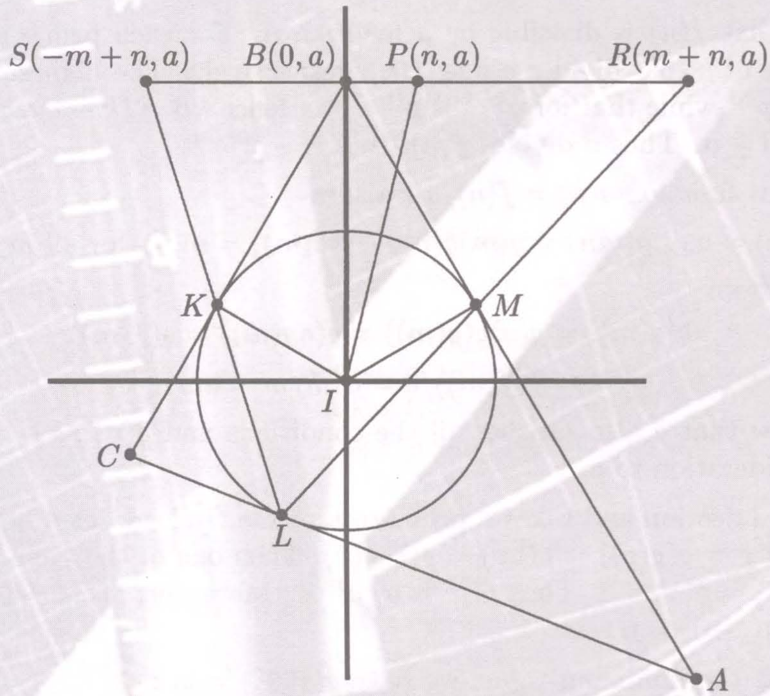
Solution: Since $ab^2 + b + 7 | b(a^2b + a + b)$ and $a^2b^2 + ab + b^2 = a(ab^2 + b + 7) + (b^2 - 7a)$, we have either $b^2 - 7a = 0$ or $b^2 - 7a$ is a multiple of $ab^2 + b + 7$. The former implies that $b = 7t$ and $a = 7t^2$. Indeed these are solutions for all positive t.

For the second case, we note that $b^2 - 7a < ab^2 + b + 7$. Thus $b^2 - 7a < 0$. For $ab^2 + b + 7$ to divide $7a - b^2$, $b = 1, 2$. The case $b = 1$ requires that $7a - 1$ be divisible by $a + 8$. The quotients are less than 7. Testing each of the possibilities yields $a = 49, 11$. These are indeed solutions.

The case $b = 2$ requires that $7a - 4$ be divisible by $4a + 11$. The quotient has to be 1 and this is clearly impossible.

5. Let I be the incentre of triangle ABC. Let the incircle of ABC touch the sides BC, CA and AB at K, L and M, respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that $\angle RIS$ is acute.

First solution: (Use coordinate geometry) Let I be the origin and the coordinates of B be $(0, a)$. Without loss of generality, assume that the inradius of ΔABC be 1. Then the coordinates of M and K are $(r, s$ and $(-r, s)$ where $r = \sqrt{a^2 - 1}/a$ and $s = 1/a$. Let the coordinates of L be (p, q). Then we have $p^2 + q^2 = 1$. Let the coordinates of R and S be (x', a) and $(x", a)$. Then $x' = [r(a - q) + p(s - a)]/(s - q) = m + n$ where $m = \sqrt{a^2 - 1}(a - q)/(1 - aq)$ and $n = p(1 - a^2)/(1 - aq)$ and $x" = -m + n$. Let P be the mid point of SR. Then the coordinates of P are (n, a) and $\angle RIS$ is acute if and only if $IP > m$ (so that P is exterior to the circle with RS as a diameter.) Now $IP^2 = a^2 + n^2 > m^2$ if and only if $(aq - 1)^2 > 0$. Thus we are done. (Note: From the proof one can conclude that result still holds if one replaces the incircle by the excircle and the incentre by the corresponding excentre.



Second solution (official): Let $\angle A = 2a$, $\angle B = 2b$ and $\angle C = 2c$. Then we have

$$\angle BMR = 90^\circ - a, \quad \angle MBR = 90^\circ - b, \quad \angle BRM = 90^\circ - c.$$

Hence $BR = BM \cos a/ \cos c$. Similarly $BS = BK \cos c/ \cos a = BL \cos a/ \cos a$. Thus

$$\begin{aligned} IR^2 + IS^2 - RS^2 &= (BI^2 + BR^2) + (BI^2 + BS^2) - (BR + BS)^2 \\ &= 2(BI^2 - BR \cdot BS) = 2(BI^2 - BK^2) = 2IK^2 > 0 \end{aligned}$$

So by the cosine law, $\angle RIS$ is acute.

6. Consider all functions f from the set $\mathbf{N}$ of all positive integers into itself satisfying

$$f\left(t^2 f(s)\right) = s\left(f(t)\right)^2,$$

for all s and t in $\mathbf{N}$. Determine the least possible value of $f(1998)$.

(Official solution): Let f be a function that satisfies the given conditions and let $f(1) = a$. By putting $s = 1$ and then $t = 1$, we have

$$f(at^2) = f(t)^2, \quad f(f(s)) = a^2 s. \quad \text{for all } s, t.$$

Thus

$$\begin{aligned} (f(s)f(t))^2 &= f(s)^2 f(at^2) = f(s^2 f(f(at^2))) \\ &= f(s^2 a^2 at^2) = f(a(ast)^2) \\ &= f(ast)^2 \end{aligned}$$

It follows that $f(ast) = f(s)f(t)$ for all s, t; in particular $f(as) = af(s)$ and so

$$af(st) = f(s)f(t) \quad \text{for all } s, t.$$

From this it follows by induction that

$$f(t)^k = a^{k-1} f(t^k), \qquad \text{for all } t, k.$$

We next prove that $f(n)$ is divisible by a for each n. For each prime p, let p^α and p^β be highest power of p that divides a and $f(n)$, respectively. The highest power of p that divides $f(n)^k$ is $p^{k\beta}$ while that for a^{k-1} is $p^{(k-1)\alpha}$. Hence $k\beta \geq (k-1)\alpha$ for all k which is possible only if $\beta \geq \alpha$. Thus a divides $f(n)$.

Thus the new function $g(n) = f(n)/a$ satisfies

$$g(a) = a, \quad g(mn) = g(m)g(n), \quad g(g(m)) = m, \qquad \text{for all } m, n.$$

The last follows from

$$\begin{aligned} ag(g(m)) &= g(a)g(g(m)) = g(ag(m)) = g(f(m)) \\ &= f(f(m))/a = a^2 m/m = am \end{aligned}$$

It is easy to show that g also satisfies all the conditions and $g(n) \leq f(n)$. Thus we can restrict our consideration to g.

Now g is an injection and takes a prime to a prime. Indeed, let p be a prime and let $g(p) = uv$. Then $p = g(g(p)) = g(uv) = g(u)g(v)$. Thus one of the factors, say $g(u) = 1$. Then $u = g(g(u)) = g(1) = 1$. Thus $g(p)$ is a prime. Moreover, $g(m) = g(n)$ implies that $m = g(g(m)) = g(g(n)) = n$.

To determine the minimum value, we have $g(1998) = g(2 \cdot 3^3 \cdot 37) = g(2)g(3)^3 g(37)$. Thus a lower bound for $g(1998)$ is $2^3 \cdot 3 \cdot 5 = 120$. There is also a g with $g(1998) = 120$. This is obtained by defining $g(3) = 2, g(2) = 3, g(5) = 37, g(37) = 5$, and $g(p) = p$ for all other primes.