A Geometrical Derivation of the Formula

$$\frac{d \tan \theta}{d\theta} = \sec^2 \theta$$

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In higher mathematics, most formulas for derivatives of trigonometric functions are proved either by using a direct method according to the definition of derivatives or by using an indirect method according to the operation rules of derivatives [1]. These methods appear to be dull and inflexible to readers. In this article, a geometrical method is given to derive the formula

$$\frac{d\tan\theta}{d\theta} = \sec^2\theta .$$

Let's consider the geometry shown in Figure 1. Unit circle O is put in the Cartesian plane. The center of the unit circle O is located at the origin O and the circle O intersects the positive x-axis Ox at point D. The tangent line AD of the unit circle O is parallel to the y-axis with the contact point at D. Radial line OA intersects AD at point A with an angle θ with respect to the positive x-axis Ox. Assuming an increment in θ is $\Delta\theta$ (<<1), the radial line OA coincides with the radial line OC which intersects AD at point C and the increment in y is $\overline{AC}$ denoted by Δy.

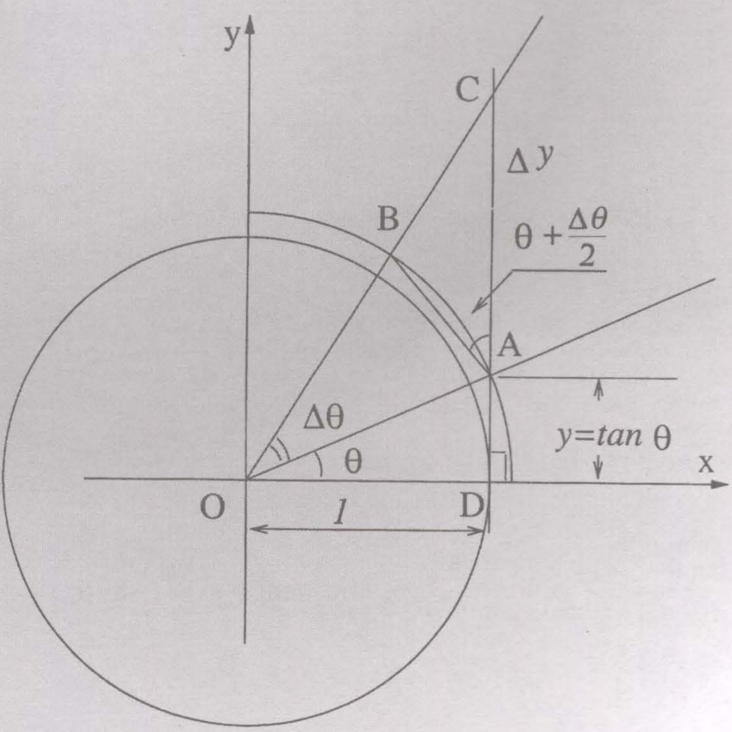

Figure 1. Geometry under consideration

In order to connect the unknown Δy with all the known data, we draw another arc AB with its center at O and radius equal to $\overline{OA}$. The arc AB intersects OA and OC at A and B, respectively.

From ΔOAD, we know

$$y = \overline{AD} = \tan\theta \tag{1}$$

and

$$\overline{OA} = \sec\theta \tag{2}$$

according to the definition of trigonometric functions. Noticing that $\Delta\theta << 1$, ΔABC can be approximated to a right triangle with $\angle ABC \approx \pi / 2$ and $\angle BAC = \theta + \frac{\Delta\theta}{2}$, and $\overline{AB}$ can be approximated as follows:

$$\overline{AB} \approx AB = \Delta\theta \cdot \overline{OA}. \tag{3}$$

Substituting (2) into (3), we have

$$\overline{AB} \approx \Delta\theta \cdot \sec\theta. \tag{4}$$

Assuming $\theta \neq (2k+1)\frac{\pi}{2}$ where k is an integer, we have $\sec\theta \neq 0$ and

$$\frac{\Delta y}{\Delta\theta} = \frac{\Delta y \cdot \sec\theta}{\Delta\theta \cdot \sec\theta} \approx \frac{\Delta y}{AB} \cdot \sec\theta = \sec\left(\theta + \frac{\Delta\theta}{2}\right) \cdot \sec\theta. \tag{5}$$

by using the express for $\overline{AB}$ in (4). Finally, we have

$$\frac{d\tan\theta}{d\theta} = \frac{dy}{d\theta} = \lim_{\Delta\theta\to 0}\frac{\Delta y}{\Delta\theta} = \lim_{\Delta\theta\to 0}\sec\left(\theta + \frac{\Delta\theta}{2}\right)\sec\theta, \tag{6}$$

i.e.,

$$\frac{d\tan\theta}{d\theta} = \sec^2\theta \tag{7}$$

which is just the formula we require. In the similar manner, one can easily prove $\frac{d\cot\theta}{d\theta} = -\csc^2\theta$.

Reference

[1] Donald Hartig, *On the Differentiation Formula for sinθ*, The American Mathematical Monthly, Vol. 96 (3) 1989, 252.