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The Qwantians & Ten Thousand Years of Slavery

Abstract

When one of us watched a class of 12 year olds tackling what we thought was a simple probability problem we didn't realise the time that we'd take to solve it. Now we can solve it by three more or less different ways, we're convinced we've seen a rich mathematical activity. But is it really a problem for 12 year olds?

The Problem

The problem is easy to state.

Apparently you have just landed on the planet Qwerty with 6 kg of super neutron fuel. Unfortunately your space craft needs 12 kg of the fuel in order to get home. Fortunately there is plenty of super neutron fuel on Qwerty. Unfortunately, the Qwertians are gamblers. They are prepared to gamble for the fuel you need. If you can throw 2, 3, 4, 10, 11 or 12 with two (fair) dice, they'll give you 2 kg of fuel. On the other hand, if you can only manage 5, 6, 7, 8 or 9, you give the Qwertians 2 kg of fuel.

There's only one more catch. If you get your 12 kg of fuel, the Qwertians are happy to give you a rousing sendoff home. If you lose all the fuel you currently have and get down to 0 kg, then, we're sorry to say, you become Qwertians slaves for 10,000 years!

So we guess the problem is, what are your chances of getting home?

Tentative Steps

The class we have on videotape got out the dice at this stage and started to simulate the problem. That's probably not a bad thing to do at the start. You'll get a feel for what the problem is about and might get some insight into the problem too. We'll leave it to you to decide what you're going to do next. You could keep reading, you could stop and see how far you can get by yourself or you could simply stop reading and go for a swim. We hope you'll try working on the problem by yourself. Even better get together with a friend and see what progress you can make. We'll still be here when you get back.

For those of you that have come on board, try a little simulation. You can do this the old-fashioned way or you can snuggle up to your friendly neighbourhood computer and set things running. The class we observed did the former. In pairs they rolled and recorded their efforts. Then they combined their results and found that, of the 56 trials they had made, 15 had been successful. So 15 times they managed to pick up 6 extra kg of fuel from the Qwertians and 41 times they ended up in slavery. Now, in the light of subsequent events, we rather think that the children cheated. Our own simulation attempts were far less "successful". Over a much longer sequence of trials, about 10,000, our chances of "escaping" appeared to be less than half those suggested by the children. Do we put this down to the optimism of youth?

So what can we say at this stage? Where are we? What progress have we made? First of all it's clear that the Qwertians are onto a good thing. They have a much better chance of picking up slaves, than you do of getting back home to Earth.

The second thing we can say, and it's probably useful to carry out this computation, is that, with every throw of the dice, the chance of getting another 2 kg of fuel is $\frac{1}{3}$ and the chance of losing 2 kg of fuel is $\frac{2}{3}$. These are not facts that the 12 year olds knew or appreciated. Some of them thought, and not unreasonably so, that because 6 numbers out of 11 gave success, that the probability of gaining fuel was $\frac{6}{11}$. If this had been the case. Qwerty would have far fewer long-serving slaves than it has right now.

A third thing that can be said is that, although we can do a few simples cases, sometimes we might be involved in a marathon dice throwing competition. To simplify things, lets write a + if we throw a 2, 3, 4, 10, 11 or 12 and a - if we throw a 5, 6, 7, 8 or 9. Then what is the "simplest case"? Surely, because we want to go home (where have we heard that before?) the best simplest case is the event + + +. If we concentrate on a good event for a while, then the next simplest cases are

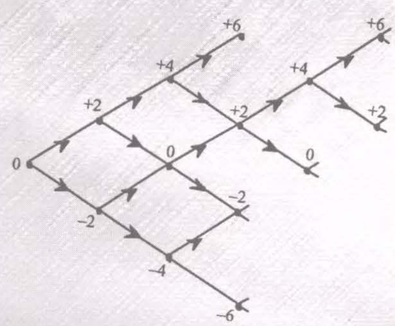
		Throw		
1st	2nd	3rd	4th	5th
−	+	+	+	+
+	−	+	+	+
+	+	−	+	+

while the next, next simplest case is

			Throw			
1st	2nd	3rd	4th	5th	6th	7th
−	−	+	+	+	+	+
−	+	−	+	+	+	+
−	+	+	−	+	+	+
−	+	+	+	−	+	+
+	−	−	+	+	+	+
+	−	+	−	+	+	+
+	−	+	+	−	+	+
+	+	−	−	+	+	+
+	+	−	+	−	+	+

But where is all that leading? How can we use this approach in any positive way when we know that we are going to have to contend with arbitrarily long sequences of pluses and minuses?

The fourth thing that comes to mind, is to do what some of the 12 year olds started to do, construct a tree diagram. Something like this, for example.



What we are clearly looking for here are all the nodes (points) with a +6 on them. We're also hoping to be able to calculate the probability of getting to such a node. And, when we've managed to do all that successfully, we want to add up all those probabilities to find out what your chances of ever leaving the planet Qwerty really are.

A hopeless task?

What does all this suggest? Maybe the tree diagram gives us some insight into how to make the + and – approach work. After all, the first "successful" node on the tree diagram comes about by throwing + + +. And the probability of doing that is $\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}$ because the probability of a + is $\frac{1}{3}$ (not $\frac{6}{11}$, for example) and the throws are independent. At least we know that the probability of escaping is $\frac{1}{27}$ + something else. At least we know your chances are better than $\frac{1}{27}$.

But then we found three ways of escaping even though we "lost" one throw. The probability of getting – + + + + is $\frac{2}{3}\times\frac{1}{3}\times\frac{1}{3}\times\frac{1}{3}=\frac{2}{81}$. But + – + + + and + + – + + also arrive with a chance of $\frac{2}{81}$ each. So the probability of escaping having had one "bad" throw is $3\times\frac{2}{81}=\frac{6}{81}$. So we now know that your chances of living at home to a ripe old age are $\frac{1}{27}+\frac{6}{81}$ + something still to be determined.

Let's take a jump here. Suppose there are f_n ways of totting up the pluses and minuses so that we get an excess of three pluses **only** at the last throw, that we throw n minuses altogether, and that we **never** have an excess of three minuses. Did you get that? We're not sure we did. The point is, no, the points are

(1) we only want to get an excess of three pluses at the last throw;
(2) we **never** get an excess of three minuses; and
(3) we throw n minuses along the way.

Just to check we're on track, we know that $f_0 = 1$ (this is the + + + case) and we know that $f_1 = 3$ (this is the – + + + +, etc, case).

What are the chances of getting "success" having thrown *n* minuses? Surely throwing ***n*** minuses has a probability of $\left(\frac{2}{3}\right)^n$?

Ah but how many pluses do we need for success? That's easy, just *n* + 3. And that has a probability of $\left(\frac{1}{3}\right)^{n+3}$.

So *n* minuses in successful throws, make a contribution of $f_n\left(\frac{1}{3}\right)^{n+3}\left(\frac{2}{3}\right)^n$ to your chances of escaping. That must mean that your chances of escaping result from summing over all *n*. Let *p* be the chances of your escaping.

We know that

$$p=\sum_{n=0}^{\infty} f_n\left(\frac{1}{3}\right)^{n+3}\left(\frac{2}{3}\right)^n$$

Hmm . But what is f_n?

Another Way?

It did occur to us earlier, when we were tinkering with the tree diagram, that there was a certain amount of repetitiveness. At the starting point, we had gained and lost no fuel. But this wasn't the only place where we were in that balanced position. And the same kind of thing was true for the +2 state. Looking back at the diagram again we can see that +2 points occur all over the tree diagram. In fact, the tree diagram only has points which are labeled 0, ±2, ±4, ±6. Can this "finiteness" be exploited in the "infiniteness" of the whole tree?

We are interested in finding *p*, right, the probability that you escape, given that you start at a zero position on the tree. What happens from the zero state? Either you gain 2 kg or you lose 2 kg of fuel. You do the first with a probability of $\frac{1}{3}$ and the second with a probability of $\frac{2}{3}$. So

$$p=\frac{1}{3}\text{(probability of escaping from a +2 point)}$$
$$+\frac{2}{3}\text{(probability of escaping from a –2 point)}$$

Call the probability in the first bracket p_{+2} and that in the second bracket p_{-2}. So

$$p=\frac{1}{3}p_{+2}+\frac{2}{3}p_{-2}.$$

Is it possible that we can get expressions for p_{+2} and p_{-2} which lead us back to *p* again? What is p_{+2}?

Surely

$$p_{+2}=\frac{1}{3}p_{+4}+\frac{2}{3}p$$

Well, that brought in another variable p_{+4}. But

$$p_4=\frac{1}{3}p_{+6}+\frac{2}{3}p_{+2}.$$

Are we getting too many variables? Can we hope to find enough equations so that we can solve them? What's on TV tonight?

See if you can solve things for yourself from here. You can then read on to see if your way is better than ours.

Now p_{+6} may cause you a problem or two. It's the probability that you escape, given that you've won 6 kg of super neutron fuel. What is this probability? Surely it's just one. In that case

$$p_{+2} = \frac{1}{3}\left(\frac{1}{3}p_{+6}+\frac{2}{3}p_{+2}\right)+\frac{2}{3}p$$
$$=\frac{1}{9}+\frac{2}{9}p_{+2}+\frac{2}{3}p.$$

If we could sort out p_{+2} we'd be in business for *p*. For what it's worth we now know that $p_{+2}=\frac{1}{7}+\frac{6}{7}p$.

Now let's go back to

$$p=\frac{1}{3}p_{+2}+\frac{2}{3}p_{-2}.$$

We can certainly eliminate p_{+2} from this equation. Can we get rid of p_{-2} too?

$$p_{-2}=\frac{1}{3}p+\frac{2}{3}p_{-4}$$
$$p_{-4}=\frac{1}{3}p_{-2}+\frac{2}{3}p_{-6}$$

Can we do anything with p_{-6}? What is the probability that you escape given that you've just lost 6 kg of fuel? No way José! There's only one conclusion: $p_{-6}=0$.

But then we're in business:

$$p_{-4}=\frac{1}{3}p_{-2},$$

and

$$p_{-2}=\frac{1}{3}p+\frac{2}{3}\left(\frac{1}{3}p_{-2}\right).$$

So

$$p_{-2}=\frac{3}{7}p.$$

Now we **are** in business!

$$p=\frac{1}{3}\left(\frac{1}{7}+\frac{6}{7}p\right)+\frac{2}{3}\left(\frac{3}{7}p\right).$$

Tidying up we get

$$p=\frac{1}{21}+\frac{2}{7}p+\frac{2}{7}p$$

or

$$\frac{3}{7}p=\frac{1}{21}$$

At last! $p=\frac{1}{9}$!

The chances of your escaping are not great. They're only $\frac{1}{9}$. Sorry, we'll put our money on the Qwertians.

Footnote

We have to admit that our computer simulations gave us $p=\frac{1}{9}$ too, so it looks as if that is the answer. You can see then why we were suspicious of the 12 year olds and their $\frac{15}{56}$! Surely that's way too high. Of course with random events you can never expect to get too near the right answer with relatively few trials but 56 trials is starting to get big "enough". (The chance of getting 15 or more successful escapes from Qwerty out of 56 attempts turns out to be just less than 1 in 1000!) Is it likely that they unconsciously ignored some throws of the dice that weren't in their favour?

One final thing. We did get the following expression for p earlier:

$$p=\sum_{n=0}^{\infty}f_n\left(\frac{1}{3}\right)^{n+3}\left(\frac{2}{3}\right)^n.$$

Again if we tidy this up a bit, we get

$$p=\frac{1}{27}\sum_{n=0}^{\infty}f_n\left(\frac{2}{9}\right)^n.$$

We now know that $p=\frac{1}{9}$, so $\sum_{n=0}^{\infty}f_n\left(\frac{2}{9}\right)^n=S$ has to be 3. Does this help us to find f_n?

It's worth noting that

$$S=1+3\times\left(\frac{2}{9}\right)+9\times\left(\frac{2}{9}\right)^2+\dots.$$

We haven't calculated f_n for $n=3$, so we can only speculate what the next term is. But so far

$$S=1+\frac{2}{3}+\frac{2^2}{9}+\cdots$$
$$=1+\frac{2}{3}+\left(\frac{2}{3}\right)^2+\cdots.$$

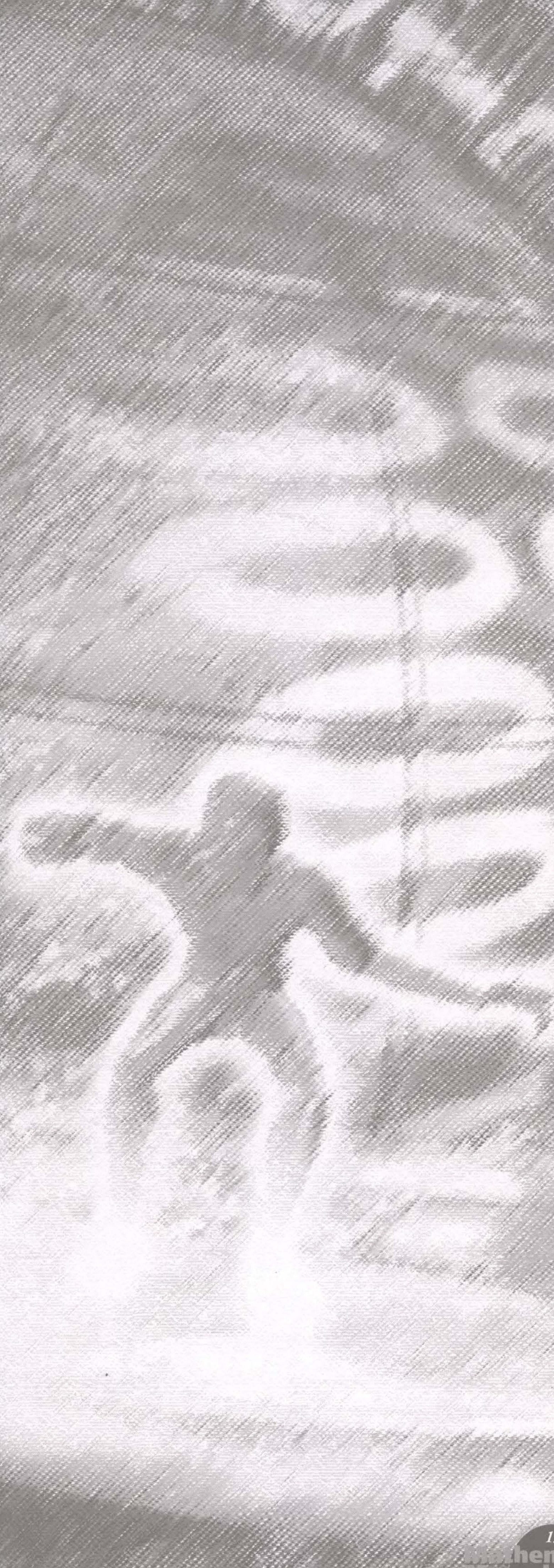
Now if the world behaved nicely (i.e. the way we wanted it to be have), then S would be a Geometric Progression with $a=1$ and $r=\frac{2}{3}$. In that case

$$S=\frac{a}{1-r}$$
$$=\frac{1}{1-2/3}$$
$$=3\ !!!$$

It might just be a GP! If it were, what would f_n be? Well $r=\frac{f_{n+1}\left(\frac{2}{9}\right)^{n+1}}{f_n\left(\frac{2}{9}\right)^n}$. That gives us $\frac{f_{n+1}}{f_n}=\frac{9}{2}\times\frac{2}{3}=3$. Could $f_n=3^n$ then?

This raises two obvious questions. First, can we show that $f_n=3^n$? If we could, then we would have found another way to get p.

But second, does f_n **have** to be 3^n? After all, we know that $S=3$ and we know that the sum of a GP with $r=\frac{2}{3}$ also sums to 3. Surely two infinite sums with the same totals **have** to have equal terms? Please!



Derek Holton is currently Professor of Pure Mathematics at the University of Otago, Dunedin, New Zealand. (That's in the south of the South Island of NZ and is close to being the nearest university to the South Pole.) His current research interests are graph theory and maths education (mainly problem solving - how it is learnt and taught). He has been team leader for the NZ Mathematical Olympiad team for a number of times. Just recently he has helped develop a web site for primary teachers which contains a lot of material on problem solving, including of the order of 100 lesson plans. His books include Mathematiccal Reflections (Springer Verlag, co authors Peter Hilton and Jean Pedersen) and Lighting Mathematical Fires (Curriculum Corporation, Melbourne).

David Fletcher is Senior Lecturer in Statistics at the University of Otago. His main research interest is in developing statistical methods to be used in ecological research. Recently he has worked on population models for assessing the conservation status of two species endemic to New Zealand: Hector's dolphin (the smallest and one of the rarest in the world) and Hooker's sealion. He is regularly consulted by government agencies to provide advice on the design and analysis of environmental data.