Some Applications of the Discriminant of a Quadratic Function

by
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Given a quadratic function $f(x)=ax^2+bx+c$, where $a \neq 0$, a, b, and c are real numbers, its discriminant, D, is defined as b^2-4ac. In this note, we will look at some nice and interesting applications of the discriminant which are normally not included in a secondary school mathematics text book.

First of all, let us look at the important properties of the discriminant.

We know that the roots of the quadratic equation $ax^2+bx+c=0$ are

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=-\frac{1}{2a}\left(b\pm\sqrt{D}\right).$$

Thus the nature of the roots which depends on D, can be summarised as follows:

Discriminant D	$D<0$	$D=0$	$D>0$
Nature of roots	Two complex roots	A pair of real and equal roots (repeated roots)	Two real and distinct roots

In addition, in the quadratic equation $ax^2+bx+c=0$, if a, b and c are rational numbers, we have:

(a) two roots are rational number if and only if D is a perfect square.

(b) two roots are irrational if and only if D is not a perfect square and $D>0$.

We can also establish two important properties of a quadratic equation as follows:

$$f(x)=ax^2+bx+c=a\left(x+\frac{b}{2a}\right)^2+\left(c-\frac{b^2}{4a}\right)=a\left(x+\frac{b}{2a}\right)^2-\left(\frac{D}{4a}\right).$$

Since $\left(x+\frac{b}{2a}\right)^2$ is always non-negative for all real x, we have:

$f(x)\geq-\frac{D}{4a}$ (i.e. f has a minimum value) if $a>0$ and

$f(x)\leq-\frac{D}{4a}$ (i.e. f has a maximum value) if $a<0$.

Thus $f(x)\geq a\left(x+\frac{b}{2a}\right)^2-\left(\frac{D}{4a}\right)>0$ for all x if and only if $D<0$ and $a>0$. Similarly, $f(x)<0$ for all real x if and only if $a<0$ and $D<0$. Hence, two very useful properties of a quadratic function can be summarised as follows.

$f(x)\geq 0$ for all real x if and only if $a>0$ and $D\leq 0$
$f(x)\leq 0$ for all real x if and only if $a<0$ and $D\leq 0$

We shall explore some applications of these two properties by using the following examples.

Finding an upper bound

Example 1: A, B and C are the interior angles of a triangle ABC.

Find an upper bound for $\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)-\cos^2\left(\frac{A+B}{2}\right)$.

Solution: We have $A+B+C=\pi$.

Let $y=-\cos^2\left(\frac{A+B}{2}\right)+\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)$.

Re-arranging, $\cos^2\left(\frac{A+B}{2}\right)-\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)+y=0$.

Treating this equation as a quadratic equation in $\cos\left(\frac{A+B}{2}\right)$ and since $\cos\left(\frac{A+B}{2}\right)$ is real, we have

$$D=\cos^2\left(\frac{A-B}{2}\right)-4y\geq 0.$$

Thus $y\leq\frac{1}{4}\cos^2\left(\frac{A-B}{2}\right)\leq\frac{1}{4}$. Hence the upper bound for $\cos\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)-\cos^2\left(\frac{A+B}{2}\right)$ is $\frac{1}{4}$. Note that with this upper bound, we can prove easily that: $\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)\leq\frac{1}{8}$, where A, B and C are anlges of a triangle.

Let $y=\sin\left(\frac{A}{2}\right)\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$.

Then $y=\frac{1}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right]\sin\left(\frac{\pi-(A+B)}{2}\right)$

$$=\frac{1}{2}\left[\cos\left(\frac{A-B}{2}\right)-\cos\left(\frac{A+B}{2}\right)\right]\cos\left(\frac{(A+B)}{2}\right)\leq\frac{1}{8}.$$

Proving inequalities

Example 2: If x, y and z are real numbers, prove that

$$x^2-xz+z^2+3y(x+y-z)\geq 0.$$

Solution: Let $f(x)=x^2-xz+z^2+3y(x+y-z)$
$$=x^2+x(3y-z)+3y(y-z)+z^2.$$

Treat f as a quadratic function in x and we check its discriminant.

$$D=(3y-z)^2-4(3y^2-3yz+z^2)$$
$$=-3y^2+6yz-3z^2$$
$$=-3(y-z)^2\leq 0.$$

As the coefficient of x^2 is 1, we can conclude that $f(x)\geq 0$ for all real x. Hence, $x^2-xz+z^2+3y(x+y-z)\geq 0$.

Determining the nature of a triangle

Example 3: *A, B* and *C* are the interior angles of a triangle *ABC*. If $\cot A+\cot B+\cot C=\sqrt{3}$, determine the nature of this triangle.

Solution: We have

$$\cot C=\cot(\pi-(A+B))=-\cot(A+B)=\frac{1-\cot A\cot B}{\cot A+\cot B}.$$

Substitute into the given condition, we have,

$$\cot A+\cot B+\frac{1-\cot A\cot B}{\cot A+\cot B}=\sqrt{3}.$$

Let $a=\cot A$, $b=\cot B$ and $c=\cot C$ and then *a, b* and *c* are real numbers. We have:

$$a+b+\frac{1-ab}{a+b}=\sqrt{3},$$

$$a^2+(b-\sqrt{3})a+(b^2-\sqrt{3}b+1)=0.$$

This is a quadratic equation in a and as a is real, its discriminant must be non-negative. Now

$$\begin{aligned}D&=(b-\sqrt{3})^2-4(b^2-\sqrt{3}b+1)\\&=-3b^2+2\sqrt{3}b-1=-(\sqrt{3}b-1)^2.\end{aligned}$$

Hence we must have $(\sqrt{3}b-1)^2=0$. Thus $b=\frac{1}{\sqrt{3}}$. As (1) is symmetric in *a* and *b*, we should have $a=\frac{1}{\sqrt{3}}$ also. Hence, $A=B=\frac{\pi}{3}$ and triangle *ABC* is equilateral.

Solving Equations

Example 4: (1983 Suzhou Secondary Schools Mathematics Competition)

Find real *x* such that $A=\frac{x^2-2x+4}{x^2-3x+3}$ is an integer.

Solution: $A=\frac{x^2-2x+4}{x^2-3x+3}=1+\frac{x+1}{x^2-3x+3}$. We need to find real *x* such that $a=\frac{x+1}{x^2-3x+3}$ is an integer. Cross multiplying, we have
$ax^2-(3a+1)x+3a-1=0$. As *x* is real, $D=(3a+1)^2-4a(3a-1)\geq 0$, so $3a^2-10a-1\leq 0$. This gives $\frac{5-2\sqrt{7}}{3}\leq a\leq\frac{5+2\sqrt{7}}{3}$. Since *a* is an integer, *a* can only take values 0, 1, 2 or 3. Substituting the values of a back into the above quadratic equation, we can solve for *x* which is -1, $2\pm\sqrt{2}$, $\frac{5}{2}$ and $\frac{7}{2}$. We can check that these *x* values produce an integer *A*.

Determining the nature of roots

Example 5: Suppose that a quadratic equation $ax^2+bx+c=0$ has real roots. Show that if *a, b* and *c* are odd, then the roots are irrational.

Proof: It suffices to prove that *D* is not a perfect square. As *a, b* and *c* are given to be odd, so $D=b^2-4ac$ is an odd number. The square of an odd number is of the form $8k+1$ as $(2n+1)^2=4n^2+4n+1=4n(n+1)+1=8k+1$.

Let $a=2m+1$, $b=2n+1$ and $c=2r+1$,
so

$$\begin{aligned}D&=(2n+1)^2-4(2m+1)(2r+1)\\&=4n(n+1)+1-4(4mr+2(m+r)+1)\\&=8\left[\frac{n(n+1)}{2}-2mr-(m+r)\right]-3\end{aligned}$$

which is not in the form of $8k+1$. Hence shown.

Conclusion

This note has shown that the discriminant can be a very useful tool in solving some mathematics problems. We hope that these examples will inspire the students to better understand and apply the discriminant and its properties.

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