PROBLEMS CORNER

CONTEST

1. Prizes in the form of book vouchers will be awarded to the first received best solution(s) submitted by secondary school or junior college students in Singapore for each of these problems.

2. To qualify, secondary school or junior college students must include their full name, home address, telephone number, the name of their school and the class they are in, together with their solutions.

3. Solutions should be typed and sent to : The Editor, Mathematics Medley, c/o Department of Mathematics, National University of Singapore, Kent Ridge, Singapore119260 ; and should arrive before 30 September 1999.

4. The Editor's decision will be final and no correspondence will be entertained.

PROBLEM 1

Let *M* denote the mid-point of the side *BC* in a triangle *ABC*. A straight line intersects *AB, AM, AC* at *D, E, F* respectively where *D* lies between *A* and *B* and *F* lies between *A* and *C*. Prove that

$$\frac{AM}{AE}=\frac{1}{2}\left(\frac{AC}{AF}+\frac{AB}{AD}\right).$$

PRIZE

One $100 book voucher

PROBLEM 2

Take any 1999 real numbers $x_1, x_2, \cdots, x_{1999}$ such that $0 \le x_n \le 1$ for all $n = 1,\ldots,1999$. Prove that

$$\left(\frac{1}{1999}\sum_{n=1}^{1999} x_n^2\right)-\left(\frac{1}{1999}\sum_{n=1}^{1999} x_n\right)^2 \le \frac{999000}{1999^2}$$

and determine when will equality hold.

PRIZE

One $100 book voucher

Solutions

Solutions to the problems in volume 25 No. 2 March 1999

PROBLEM 1

If n is a positive integer and $1 + 3^n + 9^n$ is a prime number, prove that there exists a nonnegative integer k such that $n = 3^k$.

PRIZE

One $100 book voucher

Solution to PROBLEM 1

by Lu Shang Yi
Raffles Junior College
• Class 2S01C.

We start by proving the following

<u>Lemma</u> $x^2+x+1 \mid x^{6s+2r}+x^{3s+r}+1$ for all $s \in \aleph$, $r \in \{1,2\}$.

To prove the lemma, it suffices to show that the polynomials

$p(x)=x^{6s+2r}+x^{3s+r}+1,\ q(x)=x^{6s+4r}+x^{3s+2r}+1$

are identically zero at the roots of x^2+x+1. The roots of the polynomial x^2+x+1 are the two complex cube roots of unity ω, ω^2. But $\omega^{6s+2}=\omega^{3s+2}=\omega^2$ and $\omega^{6s+4}=\omega^{3s+1}=\omega$. Therefore $p(\omega)=\varpi^{6s+2r}+\omega^{3s+1}+1=\omega^2+\omega+1=0$ and $q(\omega)=\omega^{6s+4r}+\omega^{3s+2r}+1=\omega+\omega^2+1=0$, and similarly $p(\omega^2)=q(\omega^2)=0$.

Now we proceed to the main problem.

Let 3^k be the greatest power of 3 that divides n. Write n as $3^k(3s+r)$, $s \in \aleph$ and $r \in \{1,2\}$. Now we have

$$9^n+3^n+1=9^{3^k(3s+r)}+3^{3^k(3s+r)}+1=\left(3^{3^k}\right)^{6s+2r}+\left(3^{3^k}\right)^{3s+r}+1.$$

Because of the lemma, the latter value is divisible by $\left(3^{3^k}\right)^2+\left(3^{3^k}\right)+1$, and since this latter value is a prime greater than 1, this would imply that

$$\left(3^{3^k}\right)^2+\left(3^{3^k}\right)+1=\left(3^{3^k}\right)^{6s+2r}+\left(3^{3^k}\right)^{3s+r}+1.$$

But this would imply that $3s+r=1$, and since $s \in \aleph$, $r \in \{1,2\}$, we must have $s=0$ and $r=1$ and thus $n=3^k$, a power of 3.

Solved also by Hue Yousi Daniel, Raffles Junior College, Class 2S02A. One incorrect solution was received.

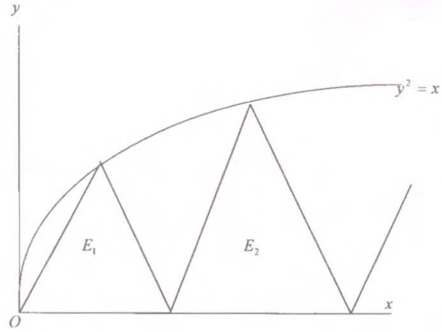
Editor's note:
The prize went to Lu Shang Yi.

PROBLEM 2

Let S denote the region which is bounded below by the x-axis and bounded above by the parabola $y^2 = x$. A sequence of equilateral triangles E_1, E_2, ... is constructed in S starting from the origin O as shown in the diagram. Find the perimeter of E_{1998}.

PRIZE

One $100 book voucher



Solution to PROBLEM 2

by Hue Yousi Daniel
Raffles Junior College
• Class 2S02A.

Answer: 3996

We let $x_0, x_1, \ldots$ denote the vertices of the equilateral triangles on the x-axis, with $x_0 = 0$, x_1 being the next vertex on the right, and so on. Let $E_1, E_2, \ldots$ denote the equilateral triangles, counting from the left, with E_1 having the base of $(x_1 - x_0)$ and so on. The x-coordinate of the point of intersection of the triangle E_n with the parabola $x = y^2$ is the solution of x in the simultaneous equations

$$y = \sqrt{3}\,(x - x_{n-1}) \qquad (1)$$

and

$$x = y^2. \qquad (2)$$

(Note that there are two solutions in x and we only take the one which gives a positive value of y in (1))

The same x also equals to

$$\frac{1}{2}(x_{n-1} + x_n) \qquad (3)$$

Solving for x in equations (1) and (2) and subsequently substituting (3) into this expression, we arrive at

$$x_n = x_{n-1} + \frac{1}{3} + \frac{1}{3}\sqrt{(12x_{n-1} + 1)} \qquad (4)$$

Since $x_0 = 0$, we calculate the subsequent values for x_i, $i = 1, 2, \ldots$ and discover that the expression within the surd, $(12x_{n-1} + 1)$, covers all the odd squares, and hence conjecture that $12x_n + 1 = (2n+1)^2$, which simplifies to give

$$x_n = \frac{1}{3}n(n+1) \qquad (5)$$

We proceed to prove equation (5) by induction. When $n = 1$, (5) is true by direct calculation. Suppose (5) is true for $n = k$. Then using (4), we have, for $n = k + 1$,

$$\begin{aligned} x_{k+1} &= x_k + \frac{1}{3} + \frac{1}{3}\sqrt{(12x_k + 1)} = \frac{1}{3}k(k+1) + \frac{1}{3} + \frac{1}{3}\sqrt{[4k(k+1)+1]} \\ &= \frac{1}{3}(k^2 + k + 1 + 2k + 1) \\ &= \frac{1}{3}(k+1)(k+2). \end{aligned}$$

Hence by mathematical induction, equation (5) is true for all positive integers n. The perimeter of E_{1998} is thus given by 3 times the base = $3(x_{1998} - x_{1997})$. Using equation (5) $x_{1998} = \frac{1}{3}(1998)(1999)$ and $x_{1997} = \frac{1}{3}(1997)(1998)$. The perimeter of E_{1998} is therefore

$$3\left[\frac{1}{3}(1998)(1999) - \frac{1}{3}(1997)(1998)\right] = 3996.$$

Solved also by Lu Shang Yi, Raffles Junior College, Class 2S01C; Seow Yongli, National Junior College, Class 98S07. Two incomplete and one incorrect solutions were received.

Editor's note:
The prize was shared as follows: $40 for Hue Yousi Daniel, and $30 each for Lu Shang Yi and Seow Yongli.