# Competition Corner

*by Tay Tiong Seng*

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# Competition Corner

In this issue we publish the problems of Auckland Mathematical Olympiad 1998, selected problems of Ukrainian Mathematical Olympiad 1997 as well as problems of the 39th International Mathematical Olympiad held in July 1998 at Taiwan. I would like to thank Dr. Chua Seng Kiat, leader of Singapore team at the 1997 International Mathematical Olympiad for bringing back these problems. We also present solutions of selected problems of the Byelorrussian Olympiads as well as the problems and solutions of the 38th International Mathematical Olympiad which was held in July 1996 at Argentina. Please send your solutions of the Auckland and the Ukrainian Mathematical Olympiads and the 39th International Mathematical Olympiad to me at the address given above. All correct solutions will be acknowledged. Note solutions that designated as official are solutions provided by the organizers of the competitions.

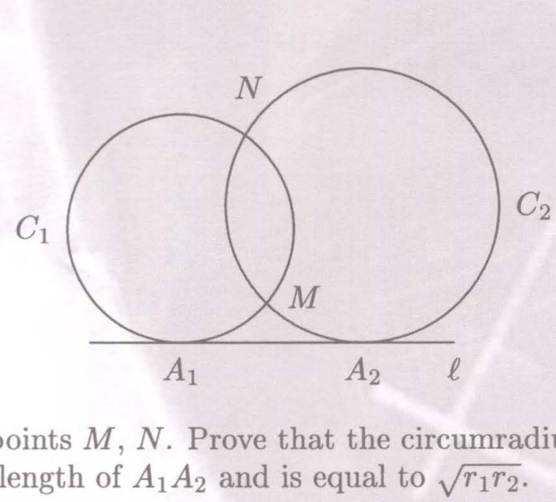
## Auckland Mathematical Olympiad 1998, Division 2

6. Find all real solutions of the system of equations

$$x + y + xy = 11$$
$$x^2 + xy + y^2 = 19$$

7. Some cells of an infinite square grid are coloured black and the rest are coloured white so that each rectangle consisting of 6 cells ($2 \times 3$ or $3 \times 2$) contains exactly 2 black cells. How many black cells might a $9 \times 11$ rectangle contain?

8. Two circles $C_1$ and $C_2$ of radii $r_1$ and $r_2$ touch a line $\ell$ at points $A_1$ and $A_2$, as shown in the figure below.




The circles intersect at points $M$, $N$. Prove that the circumradius of the triangle $A_1MA_2$ does not depend on the length of $A_1A_2$ and is equal to $\sqrt{r_1r_2}$.

9. Let $\alpha$ and $\beta$ be two acute angles such that $\sin^2\alpha + \sin^2\beta = \sin(\alpha + \beta)$. Prove that $\alpha + \beta = \pi/2$.

10. Find all prime numbers $p$ for which the number $p^2+11$ has exactly 6 different divisors (including 1 and the number itself.)

## Ukrainian Mathematical Olympiad, 1997 (Selected problems)

1. (9th grade) Consider a rectangular board in which the cells are coloured black and white alternately like chess board cells. In each cell an integer is written. It is known that the sum of the numbers in every row and every column is even. Prove that sum of all numbers in the black cells is even.

2. (10th grade) Solve the system of equations in real numbers:

$$x_1 + x_2 + \cdots + x_{1997} = 1997$$
$$x_1^4 + x_2^4 + \cdots + x_{1997}^4 = x_1^3 + x_2^3 + \cdots + x_{1997}^3.$$

3. (10th grade) Let $d(n)$ denote the greatest odd divisor of the natural number $n$. Define the function $f : \mathbb{N} \to \mathbb{N}$ by $f(2n-1) = 2^n$, $f(2n) = n + 2n/d(n)$ for all $n \in \mathbb{N}$. Find all $k$ such that $f(f(\ldots(1)\ldots)) = 1997$ where $f$ is iterated $k$ times.

4. (10th grade) In the space two regular pentagons $ABCDE$ and $AEKPL$ are situated so that $\angle DAK = 60°$. Prove that the planes $(ACK)$ and $(BAL)$ are perpendicular.

5. (11th grade) It is known that the equation $ax^3 + bx^2 + cx + d = 0$ with respect to $x$ has three distinct real roots. How many roots does the following equation have

$$4(ax^3 + bx^2 + cx + d)(3ax + b) = (3ax^2 + 2bx + c)^2?$$

6. (11th grade) Let $\mathbb{Q}^+$ denote the set all positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$, $f(x+1) = f(x) + 1$, and $f(x^2) = (f(x))^2$.

7. (11th grade) Find the minimum value of $n$ such that in any set of $n$ integers there exist 18 integers with sum divisible by 18.

8. (11th grade) At the edges $AB$, $BC$, $CD$, $DA$ of a parallelepiped $ABCDA_1B_1C_1D_1$ (not necessarily right) points $K$, $L$, $M$, $N$, respectively, are taken. Prove that the four centres of the spheres of $A_1AKN$, $B_1BKL$, $C_1CLM$, $D_1DMN$ are vertices of a parallelogram.

## Problems of 39th International Mathematical Olympiad

1. In the convex quadrilateral $ABCD$, the diagonals $AC$ and $BD$ are perpendicular and the opposite sides $AB$ and $DC$ are not parallel. Suppose that the point $P$, where the perpendicular bisectors of $AB$ and $DC$ meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles $ABP$ and $CDP$ have equal areas.

2. In a competition, there are $a$ contestants and $b$ judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose $k$ is a number such that, for any two judges, their ratings coincide for at most $k$ contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

3. For any positive integer $n$, let $d(n)$ denote the number of positive divisors of $n$ (including 1 and $n$ itself). Determine all positive integers $k$ such that

$$\frac{d(n^2)}{d(n)} = k$$

for some $n$.

4. Determine all pairs $(a, b)$ of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

5. Let $I$ be the incentre of triangle $ABC$. Let the incircle of $ABC$ touch the sides $BC$, $CA$ and $AB$ at $K$, $L$ and $M$, respectively. The line through $B$ parallel to $MK$ meets the lines $LM$ and $LK$ at $R$ and $S$, respectively. Prove that $\angle RIS$ is acute.

6. Consider all functions $f$ from the set $\mathbb{N}$ of all positive integers into itself satisfying

$$f\left(t^2 f(s)\right) = s\left(f(t)\right)^2,$$

for all $s$ and $t$ in $\mathbb{N}$. Determine the least possible value of $f(1998)$.

## Solutions of the XLV Byelorrussian Olympiads 1994/95

### Category a

1. There are 20 rooms in a hotel on a sea beach. The building of the hotel has only one storey and all rooms are arranged along one side of the common corridor. The rooms are numbered by the integers from 1 to 20 consecutively. A visitor may rent either one room for two days or two neighbouring rooms for one day. The cost of a room is \$1 per day. The sea-bathing season lasts 100 days. It is known that room No. 1 was not rented at the first day and room No. 20 was not rented at the last day of the season. Prove that owners of the hotel receive at most \$1996 during the season.

*Official solution:* Draw a $100 \times 20$ table and colour the cells of this table alternately black and white. Write the name of a visitor in the cell corresponding to the day and the number of the room he rented. By the given condition, it follows that the name of each visitor appears in the white cells as many times as in the black cells. Also the $(1, 1)$ and the $(100, 20)$ cells are of the same colour, say white, and are both not marked. Consequently, at least two black cells are also empty. Thus the owners can receive at most \$1996 during the season.

2(a). After a lesson in mathematics, the $Ox$-axis and the graph of the function $y = 2^x$ were left on a blackboard but the $Oy$-axis and the scale were erased. Give a Euclidean construction (using a straight edge and compasses only) for the $Oy$-axis and the unit of the scale.

(b). Give an Euclidean construction of both axes and the unit of the scale if both axes and the scale were erased but the graph of $y = 2^x$ and a straight line parallel to the $Ox$-axis were left on the blackboard.

*Solution 2a from Tan Chong Hui, National University of Singapore.*

Choose an arbitrary point $P = (a, 2^a)$ on the graph. Construct the line $y = 2^{a+1}$ which intersects the curve at $(a+1, 2^{a+1})$. (This line is parallel to the $Ox$-axis and $P$ is equidistant from this line and the $Ox$-axis.) This gives the unit of scale. Construct the line which is one unit from and is parallel to the $Ox$-axis. Let its intersection with the curve be $A$. Then the line through $A$ and perpendicular to the $Ox$-axis is the $Oy$-axis.

(2b) *Official solution*: Draw three parallel lines so that they are perpendicular to the given line and the distances between them are all equal. Let $M$, $K$ and $L$ denote the intersection points of these lines with the graph of $y = 2^x$; let $A$ and $N$ be the projections of $K$ and $L$ onto the leftmost line. We need to construct the intersection point $B$ of $MN$ with the desired axis $Ox$. Let $a$ denote an unknown length of the segment $AB$, and $n = AN$, $m = AM$. Let the $x$-coordinates of $M$ and $K$ be equal to $t$ and $t + \delta$, respectively. Then the $x$-coordinate of $L$ is equal to $t + 2\delta$. We have $BM = 2^t$, $BA = 2^{t+\delta}$, $BN = 2^{t+2\delta}$, it follows that $BA^2 = BM \cdot BN$ or $a^2 = (a-m)(a+n)$, i.e. $a = mn/(n-m)$. In a right angled triangle $ABC$, with $\angle A = 90°$, if $AD$ is an altitude, then $BC = AB^2/BD$. Using this, we can construct a segment of length $r^2/m$ by taking $AB = r$ and $BD = m$, where $r$ is any arbitrary fixed length. Likewise, we can construct $r^2/n$. We can also construct $r^2/m - r^2/n$ and consequently, $r^2/(r^2/m - r^2/n) = mn/(n-m) = a$. Thus we can construct a segment of length $a$. If $B \in MN$ and $AB = a$, we see that $B$ must belong to the $Ox$-axis. Thus, we draw the line that passes through $B$ and is perpendicular to $MN$. This line is the desired $Ox$-axis.

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$, satisfying the equality

$$f(f(x+y)) = f(x+y) + f(x)f(y) - xy \qquad (1)$$

for all real $x$ and $y$.

*Solution by Soh Chong Kian, Raffles Junior College. Also solved by Tan Chong Hui.*

First we prove that $f(0) = 0$. Suppose that on the contrary that $f(0) = c \neq 0$. Then putting $y = 0$ in (1), we have

$$f(f(x)) = f(x) + f(x)f(0) = (1+c)f(x). \qquad (2)$$

If $f(x) = 0$ then $f(0) = 0$ contradicting our assumption. Thus $f(x) \neq 0$ for all $x$. Next putting $y = -x = c$ in (1) and then putting $x = 0, y = 0$, we have

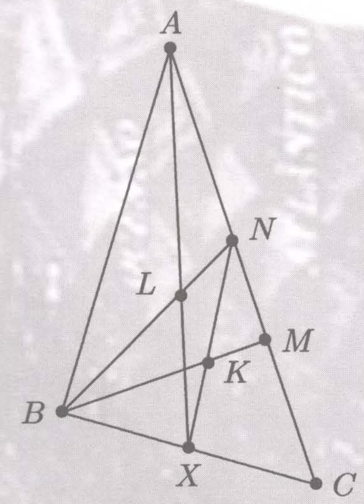
$$f(c) = c + f(c)f(-c) + c^2, \quad \text{and} \quad f(c) = c + c^2.$$

Therefore, $f(c)f(-c) = 0$ which implies $f(c) = 0$ or $f(-c) = 0$, a contradiction. Thus $f(0) = 0$.

From (2) we have $f(f(x)) = f(x)$. Thus (1) reduces to $f(x)f(y) = xy$. This gives $f(x)^2 = x^2$. If there exists $a \neq 0$ such that $f(a) = -a$, then $f(a) = f(f(a)) = f(-a)$ and $f(a)^2 = f(a)f(-a) = -a^2$. This implies that $a = 0$, a contradiction. Thus $f(x) = x$ for all $x$. It is easy to see that this function satisfies (1).

4. Given a triangle $ABC$ with $\angle ABC = 3\angle CAB$, let $M$ and $N$ be chosen on side $CA$ so that $\angle CBM = \angle MBN = \angle NBA$. Suppose that $X$ is an arbitrary point on $BC$. If $L$ is the intersection point of $AX$ and $BN$, and $K$ is the intersection point of $NX$ and $BM$, prove that the lines $KL$ and $AC$ are parallel.

*Solution by Chan Sing Chun:*




Let $\angle CAB = \theta$. Then $\angle ABC = 3\theta$ and $\angle CBM = \angle MBN = \angle NBA = \theta$. We have $\angle CNB = 2\theta$. Thus $NA = NB$, $CN = CB$. In $\Delta XBN$, since $BM$ is an angle bisector, by Stewart's Theorem $XK/KN = XB/BN$. Now consider $\Delta AXC$ with $BLN$ as a transversal, by Menelaus' Theorem

$$\frac{XL}{LA}\frac{AN}{NC}\frac{CB}{BX} = -1.$$

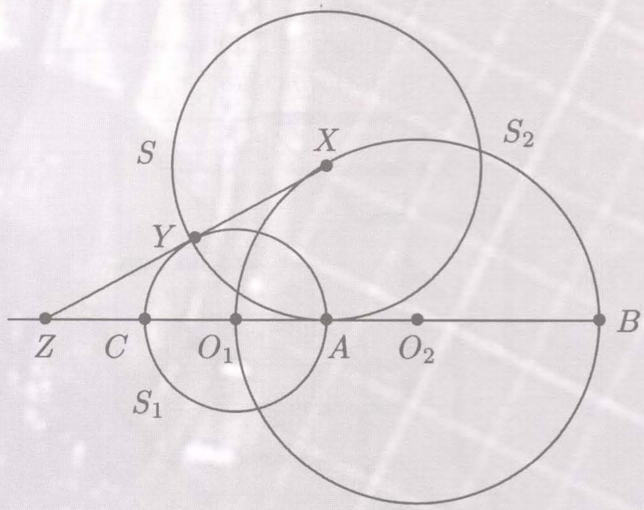
Since $CN = CB$, this reduces to $\frac{XL}{LA}\frac{AN}{XB} = 1$. Thus

$$\frac{XL}{LA} = \frac{XB}{AN} = \frac{XB}{NB} = \frac{XK}{KN}.$$

The second equality follows from $NA = NB$. The third equality follows because $XK/KN = XB/BN$ by applying Stewart's Theorem in $\Delta XBN$. Thus $KL$ is parallel to $AN$ (or $AC$).

5. The centre $O_1$ of circle $S_1$ lies on a circle $S_2$ with the centre $O_2$. The radius of $S_2$ is greater than that of $S_1$. Let $A$ be the intersection point of $S_1$ and $O_1O_2$. Consider a circle $S$ centred at an arbitrary point $X$ on $S_2$ and passing through $A$; let $Y$ be the intersection point of $S$ and $S_2$ (different from $A$). Prove that all lines $XY$ are concurrent, as $X$ runs along $S_2$.

*Official solution*: Let $r$ and $R$ denote the radii of circles $S_1$ and $S_2$, respectively. Let $X$ be an arbitrary point of $S_2$ and $Z$ be the intersection point of the line $XY$ and the line $O_1O_2$. Since $O_1Y = O_1A$ and $XY = XA$, we see that $O_1X \perp AY$. Since $\angle O_1XB = 90°$ it follows that $BX||AY$. Thus $\Delta ZYA$ is similar to $\Delta ZXB$ and $\Delta CYA$ is similar to $\Delta O_1XB$. This gives $ZA/ZB = YA/XB = CA/O_1B = r/R$. Consequently, the position of $Z$ does not depend on the choice of $X$. Hence for all choices of $X$, the lines $XY$ pass through the fixed point $Z$.




6. Given real numbers $a$ and $b$, such that the cubic polynomial $x^3+\sqrt{3}(a-1)x^2-6ax+b$ has three real roots, prove that $|b|\le|a+1|^3$.

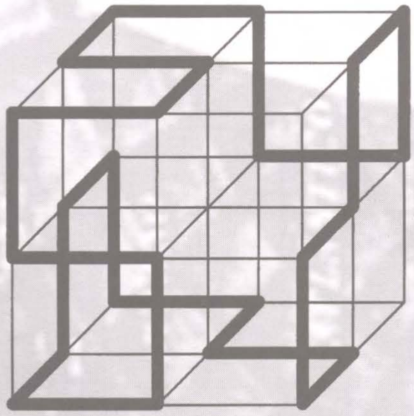
*Official solution*: Let $x_1$, $x_2$, $x_3$ be the roots of the polynomial $x^3+\sqrt{3}(a-1)x^2-6ax+b$, Then $x_1+x_2+x_3=\sqrt{3}(1-a)$, $x_1x_2+x_1x_3+x_2x_3=-6a$, $x_1x_2x_3=-b$. Thus

$$\sqrt[3]{|b|}=\sqrt[3]{|x_1x_2x_3|}\le\sqrt{\frac{x_1^2+x_2^2+x_3^2}{3}}$$
$$=\sqrt{\frac{(x_1+x_2+x_3)^2-2(x_1x_2+x_1x_3+x_2x_3)}{3}}=\sqrt{\frac{3(1-a)^2+12a}{3}}=|a+1|.$$

and hence $|b|\le|a+1|^3$.

7. The lattice frame construction of $2\times2\times2$ cube is formed with 54 metal shafts of length 1 (points of shafts' connection are called junctions). An ant starts from some junction $A$ and creeps along the shafts in accordance with the following rule: when the ant reaches the next junction it turns to a perpendicular shaft. At some moment the ant reaches the initial junction $A$; there is no junction (except for $A$) where the ant has been twice. What is the maximum length of the ant's path?

*Official solution*: The maximum length of an ant's path is equal to 24. First we prove that the path along which the ant creeps, has at most 24 junctions of the shafts of the cube frame. By the condition, any two consecutive shafts in the path (except possibly for the first and the last shafts) are perpendicular. In particular, the ant's path passes at most one shaft on every edge of the cube. Thus there are at most 12 shafts along the edges of the cube. However, each vertex in the path requires two shafts. Thus the path misses at least two vertices of the cube. Hence the ant's path passes through at most 25 junctions of the shafts. The ant's path consists of an even number of junctions. This is easily seen to be true by taking the starting point as the origin and the three mutually perpendicular lines passing through it as the axes and assume that each shaft is of unit length. Then each move by the ant causes exactly a change of one unit in one of the coordinates. Thus the total number of moves is even. Thus the length of the path is at most 24. A path of length 24 is shown in the figure.



8. Is it possible to partition the set of all rational numbers into two disjoint subsets $A$ and $B$ so that

(a) the sum of any two numbers from $A$, as well the sum of any two numbers from $B$, belongs to $A$?

(b) the sum of any two distinct numbers from $A$, as well as the sum of any two distinct numbers from $B$, belongs to $A$?

*Official solution*: The answer is no for both parts. For part (a), suppose on the contrary, that the required partition exists: $\mathbb{Q} = A \cup B$, where $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$. Let $x$ be an arbitrary element of $\mathbb{Q}$. Then we see that $x \in A$, because if $x/2 \in A$, then $x/2 + x/2 = x \in A$, and if $x/2 \in B$, then $x/2 + x/2 = x \in A$. Hence $A = \mathbb{Q}$ and $B = \emptyset$, a contradiction.

(b) Suppose the the required partition exists: $\mathbb{Q} = A \cup B$, where $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$. First we show that $B \neq \{0\}$. Indeed, if $B = \{0\}$, then $1 \in A$, $-1 \in A$, and $1 + (-1) = 0 \in A$ which is impossible. If $x \in B$ and $x \neq 0$, then one of the numbers $x/3$, $2x/3$ belongs to $B$ and the other belongs to $A$ (if both numbers belong to the same subset, then their sum $x$ would belong to $A$). We have two cases:

1) $x/3 \in B$ and $2x/3 \in A$. Then $x + x/3 = 4x/3 \in A$. Hence $2x/3 + 4x/3 = 2x \in A$.

2) $x/3 \in A$ and $2x/3 \in B$. Then $x + 2x/3 = 5x/3 \in A$. Hence $5x/3 + x/3 = 2x \in A$.

Thus $2x \in A$. Moreover, $x/2 \in A$ for if $x/2 \in B$, then $2 \cdot x/2 = x \in A$, which is a contradiction. Finally $4x \in A$, because

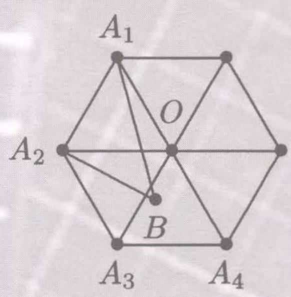
$$\begin{aligned} x/2 \in A, 2x \in A &\Rightarrow x/2 + 2x = 5x/2 \in A \\ &\Rightarrow x/2 + 5x/2 = 3x \in A \\ &\Rightarrow x/2 + 3x = 7x/2 \in A \\ &\Rightarrow x/2 + 7x/2 = 4x \in A. \end{aligned}$$

Thus we conclude that if $x \in B$ and $x \neq 0$, then $4x \in A$. Next we prove that if $x \in A$, then $4x \in A$. There are two cases: 1) $2x \in B$; 2) $2x \in A$. If $2x \in B$, then according to the above proof, we have $2 \cdot 2x = 4x \in A$. If $2x \in A$, then $2x + x = 3x \in A$ and $3x + x = 4x \in A$. Thus for any $x \in \mathbb{Q} - \{0\}$, $x/4$ is either in $A$ or $B$. In both case we have $x \in A$. Therefore, $B = \{0\}$ which is impossible. This contradiction proves the required statement.

### Category b

1. A point $B$ is marked inside a regular hexagon $A_1A_2A_3A_4A_5A_6$ so that $\angle A_2A_1B = \angle A_4A_3B = 50°$. Find $\angle A_1A_2B$.

*Official solution*: Each interior angle of the hexagon is 120°. Since $\angle A_3BA_1 = 360° - \angle BA_3A_2 - \angle BA_1A_2 - \angle A_1A_2A_3 = 120°$, we see that $B$ lies on a circle with centre $A_2$ and radius $A_2O$, where $O$ is the centre of the hexagon. Thus $A_1A_2 = BA_2$ and $\angle A_1A_2B = 180° - \angle BA_1A_2 - \angle A_1BA_2 = 80°$.




2. Find the product of three distinct real numbers $a$, $b$, $c$ if they satisfy the system of equations

$$a^3 = 3b^2 + 3c^2 - 25,$$
$$b^3 = 3c^2 + 3a^2 - 25,$$
$$c^3 = 3a^2 + 3b^2 - 25.$$

*Official solution*: Let $a$, $b$, $c$ be roots of the polynomial $f(x) = x^3 - \alpha x^2 + \beta x - \gamma$. Then $\gamma = abc$, $\beta = ab + bc + ac$ and $\alpha = a + b + c$. We have

$$a^3 + 3a^2 = 3(a^2 + b^2 + c^2) - 25$$
$$b^3 + 3b^2 = 3(a^2 + b^2 + c^2) - 25$$
$$c^3 + 3c^2 = 3(a^2 + b^2 + c^2) - 25.$$

Thus $a, b, c$ are roots of the polynomial $g(x) = x^3 + 3x^2 - 3(\alpha^2 - 2\beta) + 25$ since $a^2 + b^2 + c^2 = \alpha^2 - 2\beta$. Since $a$, $b$ and $c$ are distinct, $f(x) = g(x)$. This gives $\alpha = -3$, $\beta = 0$ and $\gamma = 3(\alpha^2 - 2\beta) - 25 = 27 - 25 = 2$, i.e., $abc = 2$.

3. "Words" are formed with the letters $A$ and $B$. Using the words $x_1, x_2, \ldots, x_n$ we can form a new word if we write these words consecutively one next to another: $x_1x_2 \ldots x_n$. A word is called a palindrome, if it is not changed after rewriting its letters in the reverse

order. Prove that any word with 1995 letters $A$ and $B$ can be formed with less than 800 palindromes.

*Official solution*: (The key idea is to find the longest word that can be formed using at most 2 palindromes.) First of all, it is easy to check that any 5-letter word may be formed with at most two palindromes. Indeed, ($A$ and $B$ are symmetric).

$AAAAA = AAAAA$, $AAAAB = AAAA + B$, $AAABA = AA + ABA$,
$AAABB = AAA + BB$, $AABAA = AABAA$, $AABAB = AA + BAB$,
$AABBA = A + ABBA$, $AABBB = AA + BBB$, $ABAAA = ABA + AA$,
$ABAAB = A + BAAB$, $ABABA = ABABA$, $ABABB = ABA + BB$,
$ABBAA = ABBA + A$, $ABBAB = ABBA + B$, $ABBBA = ABBBA$,
$ABBBB = A + BBBB$.

Let us consider an arbitrary word with 1995 letters and divide it into words with 5 letters each. Each of these $1995/5 = 399$ words may be formed with at most two palindromes. Thus any 1995-letter word may be formed with at most $399 \times 2 = 798$ palindromes.

4. Find all functions $f$, $f : \mathbb{R} \to \mathbb{R}$, satisfying the equality

$$f(f(x-y)) = f(x) - f(y) + f(x)f(y) - xy$$

for all $x$ and $y$.

*Solution*: The key is to show that $f(0) = 0$ as in Category a Problem 3.

5. Let $AK$, $BL$ and $CM$ be the altitudes of an acute angled triangle $ABC$. Prove that if $9\mathbf{AK} + 4\mathbf{BL} + 7\mathbf{CM} = \mathbf{0}$, then there is an angle in $\Delta ABC$ that is equal to $60°$.

*Official solution*: Define $p = |\mathbf{AK}|$, $q = |\mathbf{BL}|$, $r = |\mathbf{CM}|$. Then $\Delta PQR$ is similar to $\Delta ABC$, where $9p = QR$, $4q = PR$ and $7r = PQ$. Let $a$, $b$ and $c$ be the lengths of the sides of $\Delta ABC$. Since $p = 2S/a$, $q = 2S/b$ and $r = 2S/c$, where $S$ is the area of $\Delta ABC$, we have

$$a^2 = 18S/k, \quad b^2 = 8S/k, \quad c^2 = 14S/k$$

where $k = 9p/a = 4q/b = 7r/c$. Then $\angle ACB = 60°$ since by the cosine rule we have

$$\cos\angle ACB = \frac{a^2 + b^2 - c^2}{2ab} = \frac{1}{2}.$$

6. Given three real numbers such that the sum of any two of them is not equal to 1, prove that there are two numbers $x$ and $y$ such that $xy/(x+y-1)$ does not belong to the interval $(0, 1)$.

*Official solution*: Let $a$, $b$, $c$ be the given numbers. Suppose that each of the numbers

$$A = \frac{ab}{a+b-1}, \quad B = \frac{ac}{a+c-1}, \quad C = \frac{bc}{b+c-1}.$$

belongs to $(0, 1)$. Then $A > 0$, $B > 0$, $C > 0$ and

$$ABC = \frac{a^2b^2c^2}{(a+b-1)(a+c-1)(b+c-1)} > 0.$$

Hence

$$D = (a+b-1)(a+c-1)(b+c-1) > 0. \qquad (*)$$

On the other hand, $A-1<0$, $B-1<0$, $C-1<0$. Thus $(A-1)(B-1)(C-1)<0$. It is easy to verify that

$$A-1 = \frac{(a-1)(b-1)}{a+b-1}, \quad B-1 = \frac{(a-1)(c-1)}{a+c-1}, \quad C-1 = \frac{(b-1)(b-1)}{b+c-1}.$$

Consequently,

$$(A-1)(B-1)(C-1) = \frac{(a-1)^2(b-1)^2(c-1)^2}{(a+b-1)(a+c-1)(b+c-1)} < 0$$

contradicting $(*)$. This proves the statement.

7. Let $\mathbb{Q}^*$ denote the set of rational numbers, each greater than 1.

(a) Is it possible to partition $\mathbb{Q}^*$ into two disjoint sets $A$ and $B$ so that the sum of any two numbers from $A$ belongs to $A$ and the sum of any two numbers from $B$ belongs to $B$?

(b) Is it possible to partition $\mathbb{Q}^*$ into two disjoint sets $A$ and $B$ so that the product of any two numbers from $A$ belongs to $A$ and the product of any two numbers from $B$ belongs to $B$?

*Official solution*: (a) If $a \in A$ and $b \in B$, then for any $m \in \mathbb{N}$, ($\mathbb{N}$ denotes the set of natural numbers), we have $ma \in A$ and $mb \in B$. Let $p/q = a \in A$ and $r/s = b \in B$, where $p,q,r,s \in \mathbb{N}$. Then $qra = qrp/q = rp \in A$, $spb = spr/s = rp \in B$ which contradicts $A \cap B = \emptyset$. Thus it is impossible.

(b) Define $A = \{p/q | p,q \in \mathbb{N}, p > q, \gcd(p,q) = 1, q \text{ odd}\}$, $B = \mathbb{Q}^* - A$. It is obvious that sets $A$ and $B$ satisfy the problem's condition. Thus the answer is yes.

8 (a). Each side of an equilateral triangle is divided into 6 equal parts; the points of this partition are connected by lines parallel to the sides of triangle. Each vertex of the obtained triangular grid is occupied by exactly one beetle. All beetles begin crawling along the links of the grid simultaneously with the same speed. The beetles creep according to the following rule: when a beetle reaches a vertex of the grid it must turn (to the right or to the left) by 60° or 120°. (The beetles do not turn back at any point.) Prove that at some moment two beetles meet at a vertex of the grid.

(b) Would the statement remain true if each side of the triangle is divided into 5 equal parts?

*Official solution*: Let us mark 10 points as shown the figure. Consider the 10 beetles that leave these 10 points after the first move. If after the first move there are no points with more than one beetle, then there is exactly one beetle at each point. Therefore, after the first move 10 other beetles are at the marked points. After the second move these 10 beetles will be at the non-marked points. Moreover, no beetle from the set of the first 10 beetles can come back at the marked points. Consequently, we see that 20 beetles will be at non-marked points simultaneously. But there are exactly 18 non-marked points. This proves the required statement.

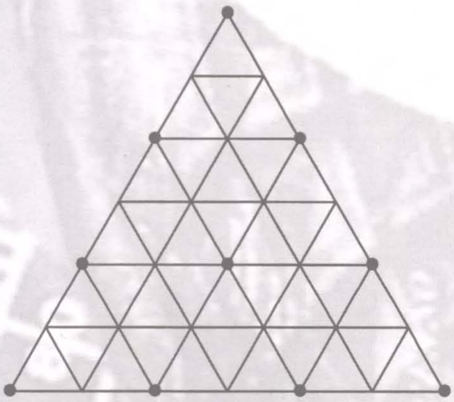


Figure 1

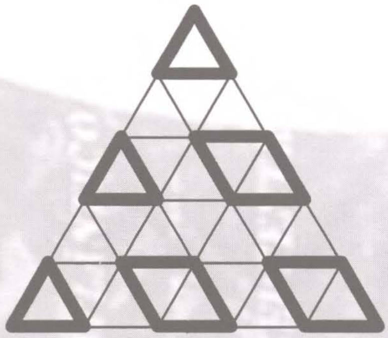


Figure 2

(b) The statement is not true. The beetles' movement is shown in Fig. 2 where they crawl along the marked triangles and parallelogram in the clockwise direction.

**Category C**

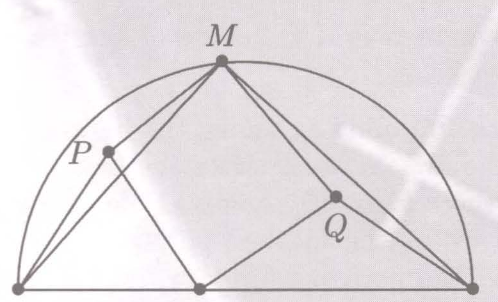
1. Six distinct numbers $n_1, n_2, n_3, n_4, n_5, n_6$ are given. For each two of these numbers, Bill calculates their sum. What is the largest possible number of distinct primes among the sums obtained by Bill?

*Solution by Chan Sing Chun:*

If the sum of two distinct numbers is a prime, then one number is odd and the other even. Since 6 distinct numbers are given, then the largest possible number of distinct primes among the sums of two numbers must come from 3 odd and 3 even numbers, Hence the largest possible number of distinct primes among the $15 = \binom{6}{2}$ different sums is $3 \times 3 = 9$. If the six numbers are $4, 8, 38, 8, 15, 33$. Then we can get 9 primes $13, 19, 37, 17, 23, 41, 47, 53, 71$.

5. Let $AB$ be the diameter of a semicircle. A point $M$ is marked on the semicircle, and a point $K$ is marked on $AB$. A circle with centre $P$ passes through $A$, $M$, $K$ and a circle with centre $Q$ passes through $M$, $K$, $B$. Prove that $M$, $K$, $P$ and $Q$ lie on the same circle.

*Solution by Chan Sing Chun:*



In the semicircle $AMB$, $\angle AMB = 90°$. Let $\angle MAB = \theta$, $\angle MBA = \lambda$. Then

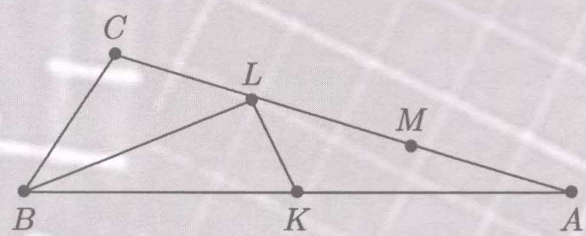
$$\theta + \lambda = 90° \tag{1}$$

$P$ is the centre of circumcircle $AKM$. Therefore $\angle MPK = 2\angle MAK = 2\theta$. $Q$ is the centre of circumcircle $MKB$. Therefore $\angle MQK = 2\angle MBK = 2\lambda$. Therefore $\angle MPK + \angle MQK = 2\theta + 2\lambda = 180°$. Thus $M, P, K, Q$ are concyclic.

**Category d**

4. Given a triangle $ABC$, let $K$ be the midpoint of side $AB$ and $L$ be a point on $AC$ such that $AL = LC + CB$. Prove that $\angle KLB = 90°$ if and only if $AC = 3CB$.

*Solution: Chan Sing Chun contributed the first part of the solution.*

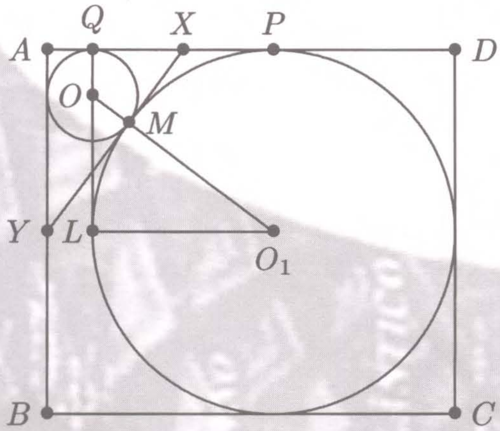



Let $M$ be the midpoint of $AL$ and $CN$ be an altitude of $\Delta CBL$. Then $KM||BL$ and $MK = BL/2$. Suppose $AC = 3CB$. Then $CL = CB$ and $NL = BL/2 = KM$. Thus $\Delta CNL$ and $\Delta LKM$ are congruent and it follows that $\angle KLB = 90°$.

Conversely, if $\angle KLB = 90°$, then $\angle LKM = 90°$, and consequently, triangles $CNL$ and $LKM$ are similar. Let $KM = a$ and $LM = b$. Then $BL = 2a$ and $BC + CL = 2b$. Thus $CL = kb$, and $NL = ka$ for some $k$, whence $MC = (2-k)b$ and $BN = (2-k)a$. Since $CNL$ and $CNB$ are right triangles, we have $BC^2 - BN^2 = CL^2 - NL^2$. This implies that $k = 1$ and $AC = 3CB$.

5. Two circles touch at a point $M$ and lie inside a rectangle $ABCD$. It is known that one of them touches the sides $AB$ and $AD$, and the other touches the sides $AD$, $BC$ and $CD$. The second circle has the radius four times as long as the radius of the first one. Find the ratios in which the common tangent of the circles that passes through $M$ divides the sides $AB$ and $AD$.

*Solution by Chan Sing Chun:*




Let the radius of the smaller circle be 1. First we find the dimension of the rectangle $ABCD$. Clearly $CD = 8$. Since $OL = 3$ and $OO_1 = 5$, we have $O_1L = 4$. This implies that $AD = 1 + 4 + 4 = 9$. Let the common tangent at $M$ cut $AD$ at $X$ and $AB$ at $Y$. Then $XP = XM = XQ$, whence $AX = 3$ and $AX/XD = 3/6 = 1/2$. Now $\tan \angle PXO_1 = PO_1/PX = 2$. Thus $\tan \angle PXM = \tan 2\angle PXO_1 = -4/3$. But $YA/AX = YA/3 = -\tan \angle PXM = 4/3$. Thus $YA = 4$ and $AY/YB = 4/4 = 1$.

6. Let $p$ and $q$ be distinct positive integers. Prove that at least one of the equations

$$x^2 + px + q = 0 \quad \text{or} \quad x^2 + qx + p = 0$$

has a real root.

*Solution by Chan Sing Chun:*

Suppose both the equations

$$x^2 + px + q = 0, \quad \text{and} \quad x^2 + qx + p = 0$$

have no real roots. Then

$$p^2 - 4q < 0, \quad \text{and} \quad q^2 - 4p < 0.$$

Thus $p^4 < 16q^2 < 64p$, whence $p < 4$. By symmetry, $q < 4$. We may also assume that $p < q$. Hence, the only possible pairs of values of $(p, q)$ are $(1, 2)$, $(1, 3)$ and $(2, 3)$. Direct checking shows the condition $q^2 - 4p < 0$ always fails.

## Solutions of the 38th International Mathematical Olympiad

1. In the plane the points with integer coordinates are the vertices of unit squares. The squares are coloured alternately black and white (as on a chessboard). For any pair of positive integers $m$ and $n$, consider a right-angled triangle whose vertices have integer coordinates and whose legs, of lengths $m$ and $n$, lie along the edges of the squares. Let $S_1$ be the total area of the black part of the triangle and $S_2$ be the total area of the white part. Let $f(m, n) = |S_1 - S_2|$.

(a) Calculate $f(m,n)$ for all positive integers $m$ and $n$ which are either both even or both odd.

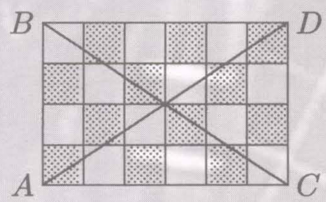
(b) Prove that $f(m,n) \le \frac{1}{2}\max\{m,n\}$ for all $m$ and $n$.

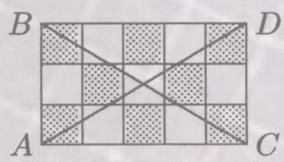
(c) Show that there is no constant $C$ such that $f(m,n) < C$ for all $m$ and $n$.

*Official solution*: (a) For an arbitrary polygon $P$, let $S_b(P)$ and $S_w(P)$ denote the total area of the white part and the black part, respectively. Let $A, B, C, D$ be the points $(0,0)$, $(0,m)$, $(n,0)$, $(n,m)$, respectively. When $m$ and $n$ are of the same parity, the colouring of the rectangle $ABCD$ is centrally symmetric about its centre. Hence $S_w(ABC) = S_w(BCD)$ and $S_b(ABC) = S_b(BCD)$. Thus

$$f(m,n) = |S_b(ABC) - S_w(ABC)| = \frac{1}{2}|S_b(ABCD) - S_w(ABCD)|.$$

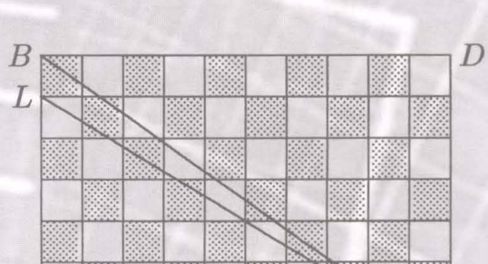
Hence $f(m,n) = 0$ when both $m$ and $n$ are even and $f(m,n) = 1/2$ when both $m$ and $n$ are odd.




(b) If $m$ and $n$ are of the same parity the result follows from (a). Now suppose that $m$ is odd and $n$ is even. Let $L$ be the point $(0, m-1)$. Then $f(m-1,n) = 0$, i.e. $S_b(ALC) = S_w(ALC)$. Thus

$$\begin{aligned} f(m,n) &= |S_b(ABC) - S_w(ABC)| = |S_b(LBC) - S_w(LBC)| \\ &\le \text{Area}\,(LBC) = \frac{n}{2} \le \frac{1}{2}\max\{m,n\}. \end{aligned}$$




(c) As in (b), with $m$ replaced by $2k+1$ and $n$ by $2k$, we have

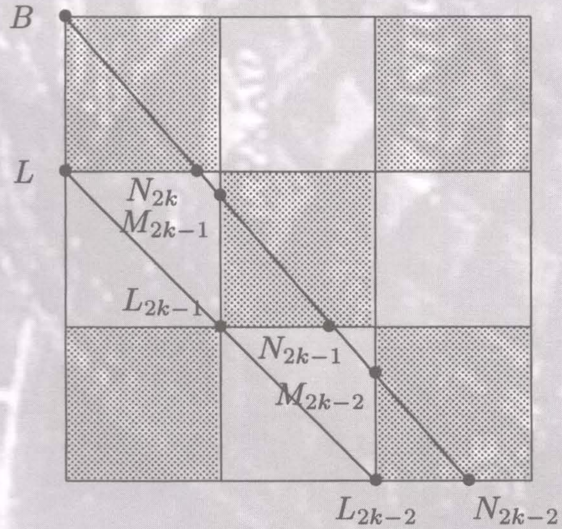
$$f(2k+1, 2k) = |S_b(LBC) - S_w(LBC)|.$$

The area of $LBC$ is $k$. Without loss of generality suppose that the hypotenuse $LC$ passes through white squares. Then the black part of $LBC$ consists of several triangles $BLN_{2k}$,

$M_{2k-1}L_{2k-1}N_{2k-1}$, ..., $M_1L_1N_1$, each of them being similar to $BAC$. We have $LN_{2k} = 2k/(2k+1)$, $BL = 1$, $M_{2k-1}L_{2k-1} = (2k-1)/2k$, $M_{2k-2}L_{2k-2} = (2k-2)/2k$, and so on. Thus their total area is

$$S_b(LBC) = \frac{1}{2}\frac{2k}{2k+1}\left(\left(\frac{2k}{2k}\right)^2 + \left(\frac{2k-1}{2k}\right)^2 + \cdots + \left(\frac{1}{2k}\right)^2\right) = \frac{4k+1}{12}.$$

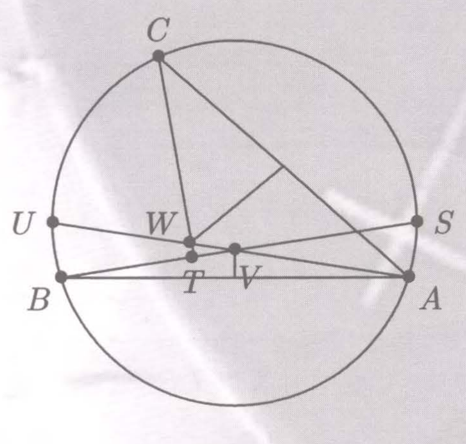
Hence $S_w(LBC) = k - \frac{4k+1}{12}$ and $f(2k+1, 2k) = (2k-1)/6$. Thus such a constant $C$ cannot exist.




2. Angle $A$ is the smallest in the triangle $ABC$. The points $B$ and $C$ divide the circumcircle of the triangle into two arcs. Let $U$ be an interior point of the arc between $B$ and $C$ which does not contain $A$. The perpendicular bisectors of $AB$ and $AC$ meet the line $AU$ at $V$ and $W$, respectively. The lines $BV$ and $CW$ meet at $T$. Show that

$$AU = TB + TC.$$

*Official solution*: (Note: The key is to notice that $AU = BS$ as in the solution.) Let the line $BV$ meet the circle at the point $S$. Then $BS = AU$. Thus we only need to prove that $TC = TS$. Let $\angle ABS = x$ and $\angle VAC = y$. Then $\angle ACS = x$, $\angle VAB = x$ and $\angle WCA = y$. Thus $\angle BSC = \angle BAC = x + y$. Also $\angle TCS = x + y$. Thus $TC = TS$ as required.



3. Let $x_1$, $x_2$, ..., $x_n$ be real numbers satisfying the conditions:

$$|x_1 + x_2 + \cdots + x_n| = 1$$

and

$$|x_i| \leq \frac{n+1}{2} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Show that there exists a permutation $y_1, y_2, \ldots, y_n$ of $x_1$, $x_2$, ..., $x_n$ such that

$$|y_1 + 2y_2 + \cdots + ny_n| \leq \frac{n+1}{2}.$$

*Solution* (Note: This has something to do with rearrangement inequality. Try to go from the smallest to the largest by a series of steps, each of which is of length $\leq n+1$.) Without loss of generality, let $x_1 + \cdots + x_n = 1$ and $x_1 \leq \cdots \leq x_n$. For each permutation $P$ of the $x_i$'s, let $S(P)$ be the corresponding sum. If $I$ is the identity permutation and $J$ is the permutation $x_n \ldots x_2 x_1$, then $S(I)$ is the largest and $S(J)$ is the smallest among all the sums. Let $r = (n+1)/2$. Then $S(I) + S(J) = 2r$. If one of $S(I)$, $S(J)$ lies between $-r$ and $r$ we are done. If not then $S(J) < -r$ and $S(I) > r$. First note that if we have a permutation $p$: $\ldots x_i x_j \ldots$ where $x_i > x_j$, and $q$ is the permutation obtained by interchanging $x_i$ and $x_j$, then $S(q) - S(p) \leq (n+1)$. We can go from $J$ to $I$ by a sequence of such operations, that is interchanging adjacent terms. Thus one of the intermediate permutations must have its sum lie between $-r$ and $r$.

4. An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$, is called a *silver* matrix if, for each $i = 1, 2, \ldots, n$, the $i$th row and $i$th column together contain all elements of $S$. Show that

(a) there is no silver matrix for $n = 1997$;

(b) silver matrices exist for infinitely many values of $n$.

*Solution* (Note: Part (a) is by a simple parity argument. The first construction is fairly standard. You should learn how to use it. The second construction is tricky and is adapted from the idea of Huah Cheng Jiann, Singapore's representative at the IMO.) (a) Let $A$ be an $n \times n$ silver matrix. For each $i = 1, 2, \ldots, n$, let $A_i$ be the set containing all the elements which are in the $i^{\text{th}}$ row and the $i^{\text{th}}$ column, excluding the diagonal element. Let $x$ be an element which is not on the diagonal. (Such an element exists because there are only $n$ entries on the diagonal but there are $2n-1$ elements.) If $x$ is at the $(i, j)$-entry, then it is in $A_i$ and $A_j$, which is called an $x$-pair. Thus $x$ partitions the sets $A_1$, ..., $A_n$ into $x$-pairs and so $n$ must be even. So there is no silver matrix of order 1997.

(b) *First construction:* Suppose $A$ is a $n \times n$ silver matrix. Construct a $2n \times 2n$ silver matrix as follows. Put two copies of $A$ on the diagonal. Then form an $n \times n$ Latin square $B$ using the symbols $2n$ to $3n-1$ (each row and each column is a permutation of the symbols.) and another, say $C$, using the symbols $3n$ to $4n-1$. Use these as the off diagonal blocks. (See the matrix below)

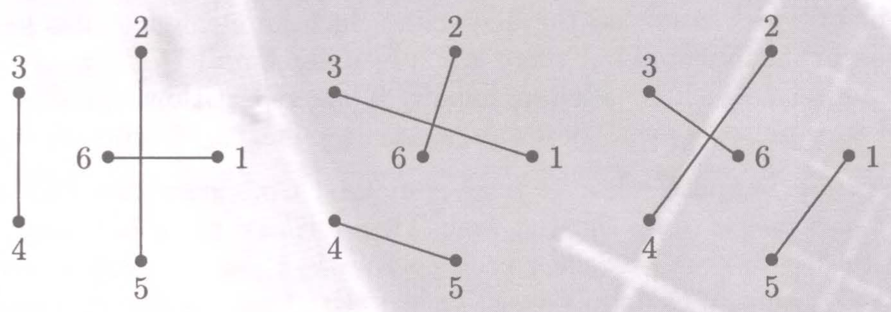
$$\begin{pmatrix} A & B \\ C & A \end{pmatrix}$$

The matrix constructed is a $2n \times 2n$ silver matrix. Starting with the $2 \times 2$ silver matrix one can construct silver matrices of order $2^n$ for any natural number $n$. An $n \times n$ Latin square can be constructed by putting $1, 2, \ldots, n$ in the first row. Each subsequent row is obtained by taking the first element of the previous row and put it at the end. The matrices below are some examples.

$$\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \rightarrow \left(\begin{array}{cc|cc} 1 & 2 & 4 & 5 \\ 3 & 1 & 5 & 4 \\ \hline 6 & 7 & 1 & 2 \\ 7 & 6 & 3 & 1 \end{array}\right) \rightarrow \left(\begin{array}{cccc|cccc} 1 & 2 & 4 & 5 & 8 & 9 & 10 & 11 \\ 3 & 1 & 5 & 4 & 9 & 10 & 11 & 8 \\ 6 & 7 & 1 & 2 & 10 & 11 & 8 & 9 \\ 7 & 6 & 3 & 1 & 11 & 8 & 9 & 10 \\ \hline 12 & 13 & 14 & 15 & 1 & 2 & 4 & 5 \\ 13 & 14 & 15 & 12 & 3 & 1 & 5 & 4 \\ 14 & 15 & 12 & 13 & 6 & 7 & 1 & 2 \\ 15 & 12 & 13 & 14 & 7 & 6 & 3 & 1 \end{array}\right)$$

*Second construction:* It can be shown that a silver matrix $A$ of order $2n$ exist for all $n$. We need a few definitions. Define a 2-partition of the set of integers $N = \{1, 2, \ldots, 2n\}$ as a division of the set into pairwise disjoint 2-element subsets whose union is $N$. For example $\{\{1,2\},\{3,4\},\{5,6\}\}$ is a 2-partition of $\{1, 2, \ldots, 6\}$. Let $a_{ij}$ denote the $(i,j)$-entry of $A$. Suppose $k$ is an element which does not appear on the main diagonal of a $2n \times 2n$ matrix. Let $X_k = \{\{i,j\} : a_{ij} = k \text{ or } a_{ji} = k\}$. Then $k \in A_i$ and $k \in A_j$ if and only if $\{i,j\} \in X_k$, where $A_i$ is as defined in part (a). Thus $k \in A_i$ for all $i$ if and only if $X_k$ is a 2-partition of $\{1, 2, \ldots, 2n\}$. If we have $(2n-1)$ 2-partitions of $N$, $B_1$, ..., $B_{2n-1}$ which are pairwise disjoint, then each pair of distinct integers $\{i,j\}$ with $1 \le i, j \le 2n$ is in exactly one of the $B_k$'s. A $2n \times 2n$ silver can be constructed as follows: Put $4n-1$ on the diagonal. For each $k = 1, 2, \ldots, 2n-1$, put $a_{ij} = k$ and $a_{ji} = k + 2n - 1$ if $i < j$ and $\{i,j\} \in B_k$. Then the resulting matrix is a silver matrix of order $2n$.

The desired 2-partitions can be constructed as follows. Consider a regular polygon with $2n-1$ sides. Label the vertices as $1, 2, \ldots, 2n-1$ and the centre of the polygon as $2n$. Let $B_k$ consists of the pair $\{k, 2n\}$ together with the pairs that are joined by lines which are perpendicular to the line joining $k$ to $2n$. Then $B_1$, ..., $B_{2n-1}$ are the desired 2-partitions.



For example, when $n = 3$, we have the 2-partitions $B_1 = \{\{1,6\},\{2,5\},\{3,4\}\}$, $B_2 = \{\{2,6\},\{1,3\},\{4,5\}\}$, $B_3 = \{\{3,6\},\{2,4\},\{1,5\}\}$, $B_4 = \{\{4,6\},\{3,5\},\{1,2\}\}$, $B_5 = \{\{5,6\},\{1,4\},\{2,3\}\}$. The picture above shows $B_1$, $B_2$ and $B_3$. Put $a_{16} = a_{25} = a_{34} = 1$,

$a_{61} = a_{52} = a_{43} = 6$, $a_{26} = a_{13} = a_{45} = 2$, $a_{62} = a_{31} = a_{54} = 7$, etc, we obtain the silver matrix

$$\begin{pmatrix} 11 & 4 & 2 & 5 & 3 & 1 \\ 9 & 11 & 5 & 3 & 1 & 2 \\ 7 & 10 & 11 & 1 & 4 & 3 \\ 10 & 8 & 6 & 11 & 2 & 4 \\ 8 & 6 & 9 & 7 & 11 & 5 \\ 6 & 7 & 8 & 9 & 10 & 11 \end{pmatrix}.$$

5. Find all pairs $(a, b)$ of integers, $a \geq 1$, $b \geq 1$ that satisfy the equation

$$a^{(b^2)} = b^a.$$

*Solution*: We need the following Lemma: If $a$, $m$, $n$ are positive integers with $m$ and $n$ coprime, and $a^{m/n}$ is also an integer, then $a = k^n$ for some positive integer $k$.

Proof of the Lemma: Let $a^m = b^n$, where $b = a^{m/n}$. Then $a$ and $b$ have same prime factors. Let $a = p_1^{c_1} p_2^{c_2} ... p_s^{c_s}$ and $b = p_1^{d_1} p_2^{d_2} ... p_s^{d_s}$. It is not hard to see that $n$ divides $c_i$ for each $i$ thus completing the proof.

For the solution of the problem, first note that $a = 1$ if and only if $b = 1$. So assume that both are not 1. Taking log, we have $b^2/a = \log b / \log a = t$. Thus $b^2 = at$ and $b = a^t$, whence $b^2 = a^{2t} = at$. Let $t = p/q$, where $p$, $q$ are coprime. Since $at$ is an integer, we have $q|a$. Moreover, if $q = 1$, then $t$ is a positive integer and $a^{2t} = at$ cannot hold. Thus $q > 1$.

First consider the case $q$ is odd. Since $a^{2p/q}$ is an integer and the pair $2p$ and $q$ are coprime, $a^{1/q}$ is an integer by the Lemma. Also $a$ is a multiple of $q$. Thus $a = (q^r k)^q$, where $r$ and $k$ are natural numbers and $q$, $k$ coprime. Thus $a^{2t} = a^{2p/q} = q^{2rp} k^{2p}$ and $at = ap/q = q^{qr-1} p k^q$, whence $q^{2rp} k^{2p} = q^{qr-1} p k^q$. Since $q > 1$, we have $2rp = qr - 1$. This implies $2p < q$. If $k > 1$, then $2p \geq q$ which leads to a contradiction. Thus $k = 1$ and $p = 1$. This gives $1 = r(q-2)$ and $r = 1$, $q = 3$, whence $a = 27$, $b = 3$.

When $q$ is even, $a^{2/q}$ is an integer. Thus $a = (q^r k)^{q/2}$, where $r$ and $k$ are natural numbers and $q$, $k$ coprime. Thus $a^{2t} = a^{2p/q} = q^{rp} k^p$ and $at = ap/q = q^{(qr/2)-1} p k^{q/2}$, whence $q^{rp} k^p = q^{(qr/2)-1} p k^{q/2}$. Using the same argument as before, we have $p = 1$, $k = 1$, $q = 4$ and $r = 1$, and the corresponding answer is $a = 16$, $b = 2$.

(Note: The key step is that $a^{2t} = at \in \mathbb{Z}$, i.e., $a^{2p/q} = ap/q \in \mathbb{Z}$. Then we can conlcude that $q$ divides $a$ and that $a$ is a $q$ power if $q$ is odd and $a$ can be written in the form given in the solution.)

6. For each positive integer $n$, let $f(n)$ denote the number of ways of representing $n$ as a sum of powers of 2 with nonnegative integer exponents. Representations which differ only in the ordering of their summands are considered to be the same. For instance, $f(4) = 4$, because the number 4 can be represented in the following four ways:

$$4; 2+2; 2+1+1; 1+1+1+1.$$

Prove that, for any integer $n \geq 3$:

$$2^{n^2/4} < f(2^n) < 2^{n^2/2}.$$

*Official solution:* Note that $2 = f(2) \leq f(n)$ for $n \geq 2$. Also, in an representation of $2n$ as a power of 2, the number 1's is always even. If there are $2k$ 1's, then the remaining terms, when divided by 2, give a representation of $n - k$. Thus

$$\begin{aligned} f(2n) &= 2 + (f(2) + \cdots + f(n)) \leq 2 + (n-1)f(n) \\ &\leq f(n) + (n-1)f(n) = nf(n), \quad \text{for } n = 2, 3, \ldots \end{aligned}$$

Consequently,

$$\begin{aligned} f(2^n) &\leq 2^{n-1} f(2^{n-1}) \leq 2^{n-1} 2^{n-2} f(2^{n-2}) \\ &\leq \cdots \leq 2^{(n-1)+(n-2)+\cdots+1} f(2) = 2^{n(n-1)/2} \cdot 2. \end{aligned}$$

And since $2^{(n(n-1)/2} \cdot 2 < 2^{n^3/2}$ for $n \geq 3$, the upper estimate follows.

To find the lower estimate we use binary representations of numbers. Let $\alpha = (a_0, a_1, \ldots)$ be a sequence of integers with only a finite number of nonzero terms. Define $S(\alpha) = a_0 2^0 + a_1 2^1 + \cdots$. For two sequences $\alpha = (a_0, a_1, \ldots)$ and $\beta = (b_0, b_1, \ldots)$, define $\alpha + \beta = (a_0 + b_0, a_1 + b_1, \ldots)$. Then $S(\alpha + \beta) = S(\alpha) + S(\beta)$. For any integer $n$ let $n = a_0 2^0 + a_1 2^1 + \cdots + a_k 2^k$ be the binary representation of $n$, i.e., $a_i = 0$, or 1 for $i = 1, 2, \ldots, k$. Let $\text{bin}(n) = (a_0, a_1, \ldots, a_k, 0, 0, \ldots)$. Then $S(\text{bin}(n)) = n$.

Now we are ready to find the lower estimate. We start by proving that

$$f(2^{n+3}) \geq 2^{2n-1} f(2^n). \qquad (*)$$

Let $\alpha = (a_0, a_1, \ldots)$ be a representation of $2^n$. We will construct $2^{2n-1}$ different representations of $2^{n+3}$. First, the sequence $\beta(\alpha) = (0, 0, a_0, a_1, \ldots)$ is a representation of $2^{n+2}$. Let $\gamma(x) = (0, x, 0, 0, \ldots) + \text{bin}(2^{n+1} - 2x)$ and $\delta(y) = (y, 0, 0, \ldots) + \text{bin}(2^{n+1} - y)$, where $0 \leq x \leq 2^n$ and $0 \leq y \leq 2^{n+1}$ are both even. Note the first entry of $\delta(y)$ is always $y$ and the first two entries of $\gamma(x)$ are always 0 and $x$. We have $S(\gamma(x)) = S(\delta(x)) = 2^{n+1}$. Define $F(\alpha, x, y) = \beta(\alpha) + \gamma(x) + \delta(y)$. Then $F(\alpha, x, y)$ is a representation of $2^{n+3}$. We need to show that $F(\alpha, x, y) \neq F(\alpha', x', y')$ if $(\alpha, x, y) \neq (\alpha', x'y')$. To this end, we let $F(\alpha, x, y) = F(\alpha', x'y')$. Then comparing the first entries, we have $y = y'$. Thus $\beta(\alpha) + \gamma(x) = \beta(\alpha') + \gamma(x')$. The second entry of the left hand side is always $x$ and that of the right hand side is always $x'$. Thus $x = x'$. This then implies $\alpha = \alpha'$. Thus the proof is complete. The inequality $(*)$ then follows readily.

We now complete the proof by induction. First we assume that for some $n > 6$ we have $f(2^n) > 2^{n^2/4}$. Then

$$f(2^{n+3}) \geq 2^{2n-1} f(2^n) > 2^{2n-1} 2^{n^2/4} \geq 2^{(n+3)^2/4}.$$

Thus the inequality holds for $n + 3$ as well. To complete the proof we need to check the cases for $n = 3, 4, \ldots, 9$. This can be done easily using the fact that $f$ is strictly increasing and the details are left to the readers.