Solution to Singapore Mathematical Olympiads 1994-1997
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Singapore Mathematical Olympiad 1994
Part A

1. We use the fact that for real numbers $a, b > 1$, $\log_a b = 1/\log_b a$. The equation above becomes

$$\log_N 2 + \log_N 3 + \log_N 4 + \cdots + \log_N 1994 = \log_N x.$$

Hence $\log_N(1994!) = \log_N x$. Therefore, $x = 1994!$.

2. Suppose the remainder when $P(x)$ is divided by $(x^2+2)(x^2+3)$ is $R(x)$. Note that the degree of $R(x)$ is ≤ 3. Since $P(x) - R(x)$ is divisible by (x^2+2), respectively, x^2+3, $P(x)$ and $R(x)$ have the same remainder when they are divided by x^2+2, respectively, x^2+3. So there are polynomials $s(x)$ and $t(x)$ such that

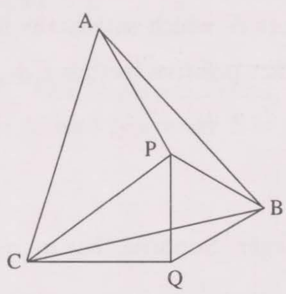
$$R(x) = s(x)(x^2+2) + 3x + 1 = t(x)(x^2+3) + 4x + 2.$$

Since $R(x)$ has degree ≤ 3, $s(x)$ and $t(x)$ have degree ≤ 1. Thus, $s(x) = ax+b$, and $t(x) = cx+d$ for some constants a, b, c and d. Then

$$\begin{aligned} (ax+b)(x^2+2) + 3x + 1 &= (cx+d)(x^2+3) + 4x + 2 \\ \Rightarrow ax^3 + bx^2 + (2a+3)x + (2b+1) &= cx^3 + dx^2 + (3c+4)x + (3d+2). \end{aligned}$$

Equating the coefficients of the various powers of x and solving, we see that $a = b = c = d = -1$. Hence $R(x) = (-x-1)(x^2+2) + 3x + 1 = -x^3 - x^2 + x - 1$.

3. Assume that $PB = 3$ cm, $PA = 4$ cm, and $PC = 5$ cm. Using B as the center, rotate the triangle ABC by $60°$ in the counterclockwise direction. Point A is rotated onto point C, and point P is rotated onto a new point Q.



Note that $PB = QB$, and $\angle PBQ = 60°$; hence triangle PBQ is equilateral. Thus $PQ = PB = 3$ cm. Since $PC = 5$ cm, $QC = PA = 4$ cm, and $(3, 4, 5)$ is a Pythagorean triple, the triangle PQC is a right triangle, with right angle at Q. Consider the triangle QBC. $QC = 4$ cm, $QB = 3$ cm, and

$$\angle BQC = \angle BQP + \angle PQC = 60° + 90° = 150°.$$

Applying the Cosine Rule to this triangle, we find that $BC^2 = 25 + 12\sqrt{3}$ cm^2. Hence, the area of triangle ABC is $\sqrt{3}\,BC^2/4 = \sqrt{3}(25 + 12\sqrt{3})/4$ cm^2.

4. First observe that the value we seek to maximize remains the same under a cyclic change of the numbers $(x_1, x_2, \ldots, x_n)$ to $(x_n, x_1, \ldots, x_{n-1})$. So by permuting the numbers cyclically, we may assume that x_n is the minimum of the numbers $x_1, \ldots, x_n$. We show that the maximum is attained when $n = 3$. For suppose we are given a collection of positive numbers $x_1, \ldots, x_n$ which add up to 1, where $n > 3$. As explained above, we may assume that $x_1 \geq x_n$. Let $y_j = x_j$ if $1 \leq j < n-1$, and $y_{n-1} = x_{n-1} + x_n$. Then

$$\begin{aligned} & y_1^2y_2 + \cdots + y_{n-2}^2y_{n-1} + y_{n-1}^2y_1 \\ = \;& x_1^2x_2 + \cdots + x_{n-3}^2x_{n-2} + x_{n-2}^2(x_{n-1} + x_n) + (x_{n-1} + x_n)^2x_1 \\ \geq \;& x_1^2x_2 + \cdots + x_{n-3}^2x_{n-2} + x_{n-2}^2x_{n-1} + x_{n-1}^2x_1 + x_n^2x_1 \\ \geq \;& x_1^2x_2 + \cdots + x_{n-3}^2x_{n-2} + x_{n-2}^2x_{n-1} + x_{n-1}^2x_n + x_n^2x_1. \end{aligned}$$

Thus it suffices to find the maximum when $n = 3$. Once again, permute the indices cyclically if necessary so that x_2 takes on the intermediate value. Then $(x_2 - x_1)(x_2 - x_3) \leq 0 \leq x_1x_2$, so

$$x_1^2x_2 + (x_2 - x_1)(x_2 - x_3)x_3 \leq x_1^2x_2 + x_1x_2x_3.$$

Rearranging, we see that

$$x_1^2x_2 + x_2^2x_3 + x_3^2x_1 \leq (x_1 + x_3)^2x_2 = (1 - x_2)^2x_2.$$

By the Arithmetic Mean-Geometric Mean inequality,

$$\frac{1 - x_2}{2} \cdot \frac{1 - x_2}{2} \cdot x_2 \leq \left(\frac{\frac{1-x_2}{2} + \frac{1-x_2}{2} + x_2}{3} \right)^3.$$

Hence

$$x_2^1x_2 + x_2^2x_3 + x_3^2x_1 \leq \frac{4}{27}.$$

Finally, observe that if $(x_1, x_2, x_3) = (2/3, 1/3, 0)$, the value of $4/27$ is attained.

5. Suppose $(N^2 - 71)/(7N + 55) = k$, where k is a positive integer. Then $N^2 - 7kN - (55k + 71) = 0$. Solving the quadratic equation for N, we see that

$$N = \frac{7k \pm \sqrt{49k^2 + 220k + 284}}{2}.$$

As N is an integer, $49k^2 + 220k + 284$ must be a perfect square. Since k is positive, by direct computation, we see that

$$(7k + 15)^2 < 49k^2 + 220k + 284 < (7k + 17)^2.$$

So we conclude that $49k^2 + 220k + 284 = (7k + 16)^2$. The only positive integer solution of this equation is $k = 7$, from which it follows that $N = 57$ or -8. One can easily check that for both of these values of N, $(N^2 - 71)/(7N + 55)$ is a positive integer.

6. Let $A = a^{2n}$, $B = b^{2n}$, and $X = x^{2n}$. Then A and B are nonnegative numbers, and we seek to maximize

$$F = \frac{(X - A)(B - X)}{(X + A)(B + X)}$$

over all nonnegative real numbers X. Computing directly,

$$F = -1 + 2(A+B)\frac{X}{(X^2+AB)+(A+B)X}.$$

Note that $X^2 + AB \geq 2\sqrt{AB}X$, and the equality is attained (at $X = \sqrt{AB}$). Therefore,

$$F \leq -1 + 2(A+B)\frac{X}{2\sqrt{AB}X + (A+B)X} = -1 + \frac{2(A+B)}{2\sqrt{AB} + (A+B)} = \left(\frac{\sqrt{A}-\sqrt{B}}{\sqrt{A}+\sqrt{B}}\right)^2.$$

Since the equality is attainable, the maximum sought is

$$\left(\frac{\sqrt{A}-\sqrt{B}}{\sqrt{A}+\sqrt{B}}\right)^2 = \left(\frac{a^n-b^n}{a^n+b^n}\right)^2.$$

7. For $1 \leq i \leq m$, let A_i be the collection of all numbers in $\{1, 2, \ldots, p^m\}$ which are divisible by p^i. Let k_i be the number of elements in A_i. Clearly, $k_i = p^{m-i}$, $1 \leq i \leq m$. We claim that

$$S(m) = k_1 + k_2 + \cdots + k_m. \tag{1}$$

To see this, suppose that $1 \leq j \leq p^m$, and $a(j) = i$. Then j is divisible by p^i but not by p^{i+1}; hence, j lies in $A_1, \ldots, A_i$, but not in $A_{i+1}, \ldots, A_m$. Therefore, the number j causes a value of i to be added to both the left and right hand sides of (1). This proves the equation. Thus

$$S(m) = k_1 + k_2 + \cdots + k_m = p^{m-1} + p^{m-2} + \cdots + p^0 = \frac{p^m - 1}{p-1}.$$

8. The only other possible arrangement is $8596 = 2 \times 14 \times 307$.

Essentially, this is found by exhaustive search. However, some observations help to cut down the number of cases that need to be checked. For example, observe that $10,000 > e \times fg \times hij > e \times f0 \times h00$. Hence $\{e, f, g\}$ can only be $\{1, 2, 3\}$ or $\{1, 2, 4\}$ (in some order). Also, the digit d is the singles digit of the product $e \times g \times j$. Thus, the singles digit of this product must be different from e, g, j or $1, 2$ (since the last two always occur in $\{e, f, g\}$).

9. Notice that f_1 is an injective function and hence has an inverse. In fact, if $g(x) = (1+x)/(2-x)$, then $g(f_1(x)) = x = f_1(g(x))$. It follows that $g(f_{n+1}(x)) = g(f_1(f_n(x))) = f_n(x)$. Now apply g to the equation $f_{35} = f_5$ five times. We see that $f_{30}(x) = x$. Apply g two more times to obtain $f_{28}(x) = g(g(x)) = 1/(1-x)$.

10. Let the area of the larger triangle be A, and that of smaller triangle be B. Then $A - B = 18$, and $A/B = k^2$ for some positive integer k. Note that

$$18 = A - B = (k^2-1)B = (k-1)(k+1)B.$$

Thus the last expression is a factorization of 18 into nonnegative integers where two of the factors ($k+1$ and $k-1$) differ by 2. It is easily checked that the only such factorization is $18 = 1 \times 3 \times 6$. From this it follows that $k - 1 = 1$, $k + 1 = 3$; so $k = 2$. Since the ratio of the areas of the triangles is k^2, the ratio of the corresponding sides is k. Therefore, the length of the corresponding side of the larger triangle is $3k = 6$ metres.

Part B

1. The answer is 'yes'. To prove the assertion, first observe that f can be factored into the form

$$f(x) = C(x-a_1)^{m_1}\cdots(x-a_j)^{m_j}(x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_k}, \qquad (2)$$

where each quadratic factor $x^2+b_rx+c_r$ cannot be factored into a product of linear (i.e., first degree) factors with real coefficients. This means that $b_r^2 - 4c_r < 0$. Therefore, by completing the squares,

$$x^2 - b_rx + c_r = \left(x - \frac{b_r}{2}\right)^2 + \left(\sqrt{c_r - \frac{b_r^2}{4}}\right)^2$$

is a sum of two squares. Now the formula

$$(A^2+B^2)(C^2+D^2) = (AC+BD)^2 + (AD-BC)^2$$

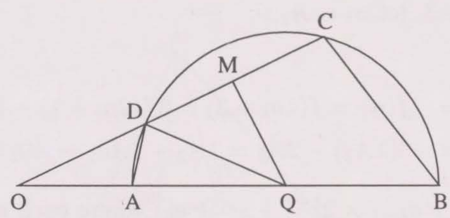
tells us that the product of two sums of two squares is also a sum of two squares. Applying this repeatedly, we see that the "quadratic" part in equation (2), namely,

$$Q(x) = (x^2+b_1x+c_1)^{n_1}\cdots(x^2+b_kx+c_k)^{n_k},$$

is a sum of two squares, say, $Q(x) = (Q_1(x))^2 + (Q_2(x))^2$. Since $f(x) \geq 0$ for all x, $C \geq 0$ and $m_1, m_2, \ldots, m_j$ are all even. Hence, $C(x-a_1)^{m_1}\cdots(x-a_j)^{m_j} = (P(x))^2$ for some real polynomial P. So we see that

$$f(x) = (P(x))^2\left((Q_1(x))^2 + (Q_2(x))^2\right) = (P(x)Q_1(x))^2 + (P(x)Q_2(x))^2.$$

2. Let Q be the center of the semicircle. Denote by r and L respectively the radius of the semicircle, and the length OQ. Here r and L are given constants.



We proceed to express the area of $ABCD$ in terms of the variable $\alpha = \angle DOA$. Let M be the foot of the perpendicular from Q to CD. Then M bisects CD. From the triangle OMQ, we see that $OM = L\cos\alpha$ and $MQ = L\sin\alpha$. Applying Pythagoras' Theorem to the triangle DMQ, we find that $DM = \sqrt{r^2 - L^2\sin^2\alpha}$.

Now

$$\begin{aligned}
\text{area } ABCD &= \text{area } OBC - \text{area } OAD \\
&= \frac{1}{2}\sin\alpha\, OB\cdot OC - \frac{1}{2}\sin\alpha\, OA\cdot OD \\
&= \frac{1}{2}\sin\alpha\Big((L+r)(L\cos\alpha + \sqrt{r^2 - L^2\sin^2\alpha}) \\
&\qquad -(L-r)(L\cos\alpha - \sqrt{r^2 - L^2\sin^2\alpha})\Big) \\
&= L\sin\alpha(\sqrt{r^2 - L^2\sin^2\alpha} + L\cos\alpha)
\end{aligned}$$

When the area of $ABCD$ is at a maximum, the derivative of this expression with respect to α is equal to 0. From this we obtain

$$(2h\cos\alpha - r)(h + r\cos\alpha) = 0,$$

where $h = \sqrt{r^2 - L^2\sin^2\alpha} = DM = CD/2$. Therefore, $r = 2h\cos\alpha = CD\cos\alpha$.

3. The definition of f (and the statement of part (i)) suggests a connection with base 2 arithmetic. By tabulating n and $f(n)$ in binary notation, we note that $f(n)$ seems to be the reversal of the binary digits of n. Let us prove that this is indeed the case by induction on n. Take as the starting point the easily verified statements

$$f(1_2) = 1_2 \quad f(10_2) = 1_2 \quad \text{and} \quad f(11_2) = 11_2.$$

Now suppose that n is an integer greater than 3, and that it has been proven that $f(m)$ is the reversal of the binary digits of m for every positive integer $m < n$. Looking at the definition of f, we see that 3 cases arise, depending on whether n has the form $2m$, $4m+1$, or $4m+3$. Let us introduce that notation $\tilde{A}$ to denote the reversal of the digits in A, if A is a string of binary digits (i.e., 0's and 1's). If n has the form $2m$, then the binary representation of n has the form $A0_2$, where the string A_2 is the binary representation of m. Now,

$$f(A0_2) = f(n) = f(2m) = f(m) = f(A_2) = \tilde{A}_2,$$

where the inductive hypothesis is used in the last equality. Similary, if $n = 4m+1$, let $m = A_2$,

$$\begin{aligned} f(A01_2) &= f(n) = f(4m+1) = 2f(2m+1) - f(m) = 2f(A1_2) - f(A_2) \\ &= 2(1\tilde{A}_2) - \tilde{A}_2 = 1\tilde{A}_2 + 1\tilde{A}_2 - \tilde{A}_2 = 10\tilde{A}_2. \end{aligned}$$

Finally, if $n = 4m+3$, let $m = A_2$,

$$\begin{aligned} f(A11_2) &= f(n) = f(4m+3) = 3f(2m+1) - 2f(m) = 3f(A1_2) - 2f(A_2) \\ &= 3(1\tilde{A}_2) - 2\tilde{A}_2 = 1\tilde{A}_2 + 1\tilde{A}0_2 - \tilde{A}0_2 = 11\tilde{A}_2. \end{aligned}$$

So, for $x = a_k \times 2^k + a_{k-1} \times 2^{k-1} + \cdots + a_0$, where each a_k is either 0 or 1,

$$f(x) = a_0 \times 2^k + a_1 \times 2^{k-1} + \cdots + a_k.$$

For part (ii), we need to count the number of palindrome binary numbers from 1 to $1994 = 11111001010_2$. (A *palindrome* is a number which is the same whether it is read forwards or backwards.) The number of $2m$-digit and $2m-1$-digit binary palindromes are both 2^{m-1}, since in each case the first digit must be 1, the next $m-1$ digits can be any combination of 0's and 1's, while the remaining digits are completely determined by the previous digits. Therefore, the total number of binary palindromes with fewer than or equal to 11 digits is

$$1 + 1 + 2 + 2 + 4 + 4 + 8 + 8 + 16 + 16 + 32 = 94.$$

Of these palindromes, only 11111011111_2 and 11111111111_2 exceed 1994. Hence the answer to part (ii) is 92.

4. We prove the assertion by induction on n. For $n = 1$, the statement is obvious. Now assume that the result holds for some $n \geq 1$. Let $x_1, x_2, \ldots, x_n, x_{n+1}$ be positive numbers satisfying $x_1 x_2 \cdots x_n x_{n+1} = 1$. We want to prove that

$$(1 + x_1 t)(1 + x_2 t) \cdots (1 + x_n t)(1 + x_{n+1} t) \geq (1 + t)^{n+1} \tag{3}$$

for all $t \geq 0$. If all the numbers $x_1, x_2, \ldots, x_n, x_{n+1}$ are equal to 1, Equation (3) obviously holds. Otherwise, at least one of the numbers $x_1, x_2, \ldots, x_n, x_{n+1}$ must be greater than 1, and at least one must be smaller than 1. Without loss of generality, let us say that $x_n > 1$, and $x_{n+1} < 1$. Then, for all $t \geq 0$,

$$\begin{aligned}(1 + x_n t)(1 + x_{n+1} t) &= 1 + (x_n + x_{n+1})t + x_n x_{n+1} t^2 \\ &\geq 1 + (1 + x_n x_{n+1})t + x_n x_{n+1} t^2 = (1 + t)(1 + x_n x_{n+1} t),\end{aligned}$$

since $x_n(1 - x_{n+1}) \geq (1 - x_{n+1})$ implies $x_n + x_{n+1} \geq 1 + x_n x_{n+1}$. Define $y_j = x_j$ for $1 \leq j < n$, and let $y_n = x_n x_{n+1}$. By the inductive assumption, for all $t \geq 0$,

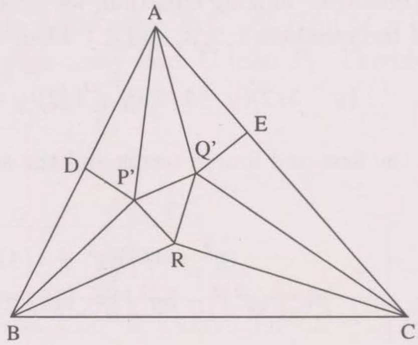
$$(1 + y_1 t)(1 + y_2 t) \cdots (1 + y_n t) \geq (1 + t)^n.$$

Therefore,

$$\begin{aligned}(1 + x_1 t) \cdots (1 + x_n t)(1 + x_{n+1} t) &\geq (1 + x_1 t) \cdots (1 + x_{n-1} t)(1 + x_n x_{n+1} t)(1 + t) \\ &= (1 + y_1 t) \cdots (1 + y_n t)(1 + t) \geq (1 + t)^{n+1}.\end{aligned}$$

This completes the induction.

5. Let a, b, and c be the sizes of the trisected angles at A, B, and C respectively. Construct the picture below so that $P'B$ and RB trisect the angle B, $Q'C$ and RC trisect the angle C, $\angle P'RB = 60° + c$, $\angle Q'CE = 60° + b$, $DB = RB$, and $EC = RC$.



By construction, $\angle P'RQ' = 60°$. Extend BP' and CQ' to meet at S. Then BR and CR are bisectors of two of the angles of the triangle BCS. Since the three angle bisectors of a triangle always meet at one point, SR bisects the angle BSC. Hence $\angle BSR = 90° - b - c$. From the triangle BSR, we calculate that

$$\angle BRS = 180° - b - (90° - b - c) = 90° + c.$$

Thus $\angle P'RS = 30°$. It follows that $\angle Q'RS = 60° - 30° = 30°$. So the triangles $P'RS$ and $Q'RS$ are congruent. In particular, $P'R = Q'R$. Since $\angle P'RQ' = 60°$, the triangle $P'RQ'$ is equilateral. To complete the proof, we need to show that $P'A$ and $Q'A$ trisect the angle A, so that P' and Q' coincide with P and Q respectively. Observe that the triangles $DP'B$ and $RP'B$ are congruent. From this we derive two consequences. The first one is that

$$DP' = P'R = P'Q'.$$

The second consequence is that $\angle DP'B = \angle BP'R = 60° + a$; hence

$$\angle DP'Q' = 360° - 2(60° + a) - 60° = 180° - 2a.$$

Similarly, $EQ' = Q'R = P'Q'$, and $\angle EQ'P' = 180° - 2a$. It follows readily that

$$\angle P'DQ' = a = \angle P'EQ'.$$

Thus a circle runs through the four points D, P', Q', and E. Suppose this circle intersects the (extended) line CE at a point A'. Since the chords DP', $P'Q'$, and $Q'E$ are equal in length, they subtend the same angle at the point A'. Notice also that $\angle DA'Q' = 180° - \angle DP'Q' = 2a$. Therefore, $\angle DA'P' = \angle P'A'Q' = \angle Q'A'E = a$. In particular, $\angle DA'E = 3a = \angle DAE$. It follows that $A = A'$. Since $P'A$ and $Q'A$ trisect the angle A, the proof is complete.

Solution to Singapore Mathematical Olympiad 1995
Part A

1. Note that each of the terms $|x - 2y|$, $(2y - 1)^2$, and $\sqrt{2z + 4x}$ are nonnegative. Therefore, their sum is 0 if and only if each term is 0. Solving the equations

$$|x - 2y| = 0, \quad (2y-1)^2 = 0 \quad \text{and} \quad \sqrt{2z + 4x} = 0$$

yield the solution $x = 1$, $y = 1/2$, and $z = -2$. Therefore, $x + y + z = -1/2$.

2. To obtain a more "symmetric" looking equation, we make the change of variable $y = x + 5/2$. (5/2 is the average of the numbers $1, 2, 3$, and 4.) Then the above equation becomes

$$(y - 3/2)(y - 1/2)(y + 1/2)(y + 3/2) = 8.$$

Multiplying together the first and fourth terms and the second and third terms on the left, we obtain

$$\begin{aligned} (y^2 - 9/4)(y^2 - 1/4) &= 8 \\ \Leftrightarrow \quad (y^2)^2 - 5y^2/2 - 119/16 &= 0. \end{aligned}$$

Solving the quadratic equation in y^2, and remembering that $y^2 \geq 0$, we get the solution $y^2 = 17/4$. Hence $y = \pm\sqrt{17}/2$. Therefore, $x = y - 5/2 = (-5 \pm \sqrt{17})/2$.

3. The value of y is $1000b + 100(a+1) + 10b + a = 1010b + 101a + 100$. Let $x = \sqrt{y}$. Then x is a positive integer, and $x^2 - 100 = 101(10b + a)$. Hence 101 divides $x^2 - 100 = (x + 10)(x - 10)$. Since 101 is a prime number, this means that either 101 divides $x + 10$ or 101 divides $x - 10$. Note that since $x^2 = y$ is a four digit number, $31 < x < 100$. Thus, $21 < x - 10 < x + 10 < 110$. The only number between 21 and 110 which is divisible by 101 is 101. So either, $x - 10 = 101$ or $x + 10 = 101$. In the first case, $x = 111 > 100$, which is impossible. In the second case, $x = 91$, and $x^2 = 8281$ fits the description of the number y. Therefore, $\sqrt{y} = 91$.

4. Certainly 3^{1000} is divisible by $3^2 = 9$. Recall that the sum of the digits of a number divisible by 9 is also divisible by 9. Hence, the numbers a, b, and c are all divisible by 9. Now the value of a is at most $9 \times 478 = 4302$. In turn, this implies that the value of b is at most $4 + 9 + 9 + 9 = 31$. Since b is divisible by 9, b must be one of the number 9, 18, or 27. In all three cases, the sum of the digits is 9. Therefore, $c = 9$.

5. Let the hundreds, tens, and ones digits of A be a, b, and c respectively. If a, b, and c are not in increasing order, rearranging them into increasing order will decrease the value of A while leaving the value of B unchanged. Hence, at the minimum, we must have $1 \leq a < b < c$. Now

$$\frac{A}{B} = \frac{100a + 10b + c}{a + b + c} = 1 + \frac{99a + 9b}{a + b + c}.$$

The fraction is minimized by making c as large as possible. Since $a < b < c$, we may take $c = 9$. Then

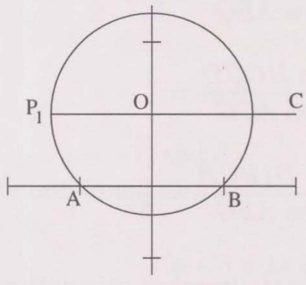
$$\frac{A}{B} = 1 + \frac{99a + 9b}{a + b + 9} = 1 + 9 + \frac{90a - 81}{a + b + 9}.$$

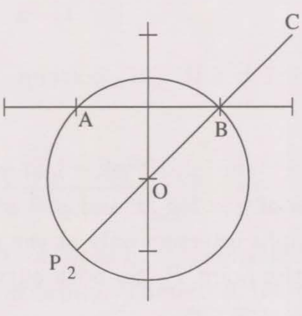
For a fixed $a \geq 1$, the last fraction is minimized by making b as large as possible. So $b = 8$. Finally,

$$\frac{A}{B} = 10 + \frac{90a - 81}{a + 17} = 10 + 90 - \frac{1611}{a + 17}.$$

This is minimized by taking $a = 1$. Therefore, the minimum value of A/B is $189/(1 + 8 + 9) = 10.5$.

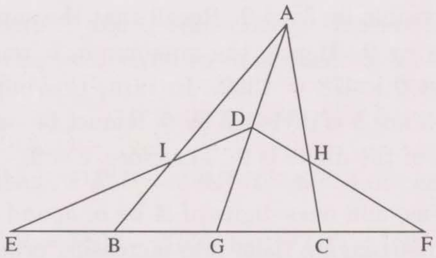
6. Obviously, P, A, B are not collinear. Let O be the center of the circle $\mathbf{O}$ which passes through these three points. Since O is equidistant from A and B, it lies on the y-axis. Moreover, $\angle AOB = 2 \times \angle APB = 90°$. Therefore, O must be either $(0, 1)$ or $(0, -1)$, and radius of $\mathbf{O} = \sqrt{2}$. Note that any point Q on $\mathbf{O}$ satisfies $\angle AQB = \frac{1}{2} \times \angle AOB = 45°$. Thus, our problem is to find the point on the circle $\mathbf{O}$ which is furthest away from C. This is the point obtained by extending the line segment CO to meet the circle on the other side of the point O. If $O = (0, 1)$, we thus obtain the point $P_1(-\sqrt{2}, 1)$; if $O = (0, -1)$, the point obtained is $P_2(-1, -2)$. Clearly, P_2 is further away from C than P_1. Therefore, $P = P_2 = (-1, -2)$.





The two cases in Problem 6

7. Consider the "left half" of the following diagram.



Since D is the midpoint of AG,

$$\text{area } ADI = \text{area } GDI. \tag{4}$$

Similarly,

$$\text{area } EBI = \text{area } GBI. \tag{5}$$

Also,

$$\text{area } DEG = 2 \times \text{area } DBG = \text{area } ABG.$$

Hence,

$$\text{area } EBI = \text{area } DEG - \text{area } IBGD = \text{area } ABG - \text{area } IBGD = \text{area } ADI. \tag{6}$$

Combining equations (4), (5), and (6), we have

$$\text{area } GDI = \text{area } ADI = \text{area } EBI = \text{area } GBI.$$

Then

$$\text{area } IBGD = \text{area } IBG + \text{area } IDG = 2 \times \text{area } ADI.$$

and

$$\text{area } ABG = \text{area } ADI + \text{area } IDG + \text{area } IBG = 3 \times \text{area } ADI.$$

Therefore,

$$\frac{\text{area } IBGD}{\text{area } ABG} = \frac{2}{3}.$$

Similarly,

$$\frac{\text{area } HCGD}{\text{area } ACG} = \frac{2}{3}.$$

Therefore,

$$\frac{\text{area } DIBCH}{\text{area } ABC} = \frac{2}{3}.$$

8. The graphs of $y = \log_a x$ and $y = a^x$ can only intersect at points on the line $y = x$. Suppose the two graphs intersect only at the point $P(c, c)$, then the line $y = x$ must be tangent to both graphs at the point P. So both curves have slope 1 at $x = c$, and $\log_a c = c = a^c$. The slope of $y = a^x$ at $x = c$ is

$$\left(\frac{d}{dx}a^x\right)_{|x=c} = \left(a^x \ln a\right)_{|x=c} = a^c \ln a.$$

Hence $\ln a = 1/a^c = 1/c$. Thus $a = e^{1/c}$. But then $c = a^c = e$. Therefore, $a = e^{1/c} = e^{1/e}$.

9. Notice that

$$1 = a+b+c+d+e+f+g \le (a+b+c)+(d+e+f)+(e+f+g) \le 3M.$$

Thus $M \ge 1/3$. The example $(a,b,c,d,e,f,g) = (1/3,0,0,1/3,0,0,1/3)$ shows that M can indeed be $1/3$. Therefore, $1/3$ is the minimum value sought.

10. Note that $xyz = 1$ implies that none of the numbers x, y, or z can be 0. Eliminate x from the expression for S by writing $x = \frac{1}{yz}$. Then

$$\begin{aligned} S &= \frac{\frac{1}{yz}+1}{\frac{1}{z}+\frac{1}{yz}+1} + \frac{y+1}{yz+y+1} + \frac{z+1}{\frac{1}{y}+z+1} \\ &= \frac{1+yz}{y+1+yz} + \frac{y+1}{yz+y+1} + \frac{yz+y}{1+yz+y} \\ &= \frac{2(1+yz+y)}{1+yz+y} = 2. \end{aligned}$$

Part B

1. From the hypothesis, we conclude that $x^3+ax^2+bx+c = (x-a)(x-b)(x-c)$. Comparing coefficients of the powers of x, we see that

$$\begin{aligned} a+b+c &= -a && (7)\\ ab+bc+ca &= b && (8)\\ -abc &= c. && (9) \end{aligned}$$

From equation (9), we arrive at two cases : $c = 0$ or $ab = -1$. If $c = 0$, then $a+b = -a$, and $ab = b$. So either $a = 0$, $b = 0$, or $a = 1$, $b = -2$.

If $ab = -1$, then neither a nor b can be 0. Eliminate b from the equations by setting $b = -1/a$. Equations (7) and (8) become

$$\begin{aligned} c &= \frac{1-2a^2}{a} && (10)\\ c(a^2-1) &= a-1. && (11) \end{aligned}$$

If $a = 1$, then $b = -1/a = -1$, and $c = -1$ from equation (10). If $a \ne 1$, equation (11) gives $c(a+1) = 1$. Therefore, from (10),

$$1 = c(a+1) = \frac{(a+1)(1-2a^2)}{a}.$$

Thus $2a^3+2a^2-1 = 0$. The only possible rational solutions of this equation are ±1 and $\pm1/2$. It is easily checked that none of these is in fact a solution. Therefore, the triple (a,b,c) must be one of $(0,0,0)$, $(1,-2,0)$, or $(1,-1,-1)$.

2. Recall that the centriod of a triangle is the intersection of its medians. Thus we are asked to show that

$$\frac{A_1B_3}{B_3A_2} = \frac{A_2B_1}{B_1A_3} = \frac{A_3B_2}{B_2A_1} = 1.$$

Let the values of the three fractions be denoted by p, q and r respectively. Label the areas of the various subtriangles as shown. Then

$$p = \frac{a}{b} = \frac{a+a+d}{b+a+c}.$$

Note that this implies that $p = (a+d)/(a+c)$. Similarly,

$$q = \frac{a}{c} = \frac{a+b}{a+d} \quad \text{and} \quad r = \frac{a}{d} = \frac{a+c}{a+b}.$$

Thus

$$pqr = \frac{a+d}{a+c}\,\frac{a+b}{a+d}\,\frac{a+c}{a+b} = 1.$$

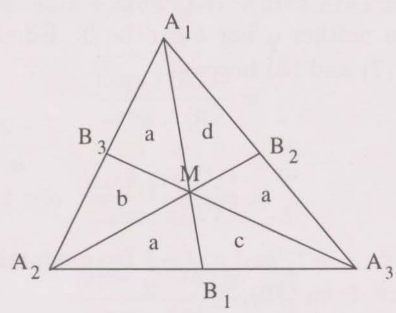
Moreover, $c = a/q$, and $d = a/r$. Hence

$$p = \frac{a+d}{a+c} = \frac{a(1+\frac{1}{r})}{a(1+\frac{1}{q})} = \frac{q(r+1)}{r(q+1)}.$$

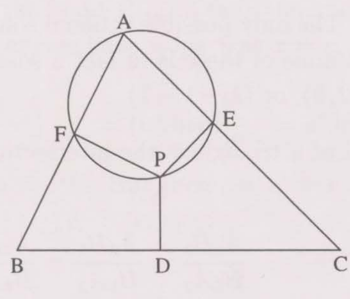
This implies that $1 + pr = pqr + pr = qr + q$. Similarly

$$1 + qp = rp + r \quad \text{and} \quad 1 + rq = pq + p.$$

Adding the three equations together, we obtain $3 = p + q + r$. Hence both the arithmetic mean and the geometric mean of the numbers p, q, r are equal to 1. This can only happen if $p = q = r = 1$, which is what we want to prove.



3. Construct a circle using AP as diameter.

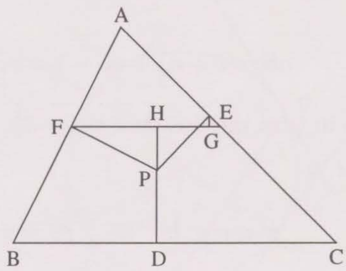


Since $\angle AFP = 90°$, F lies on the circle. Similarly, E lies on the circle as well. It follows that $\angle AEF = \angle APF$. Applying the Sine Rule to the triangles AEF and AFP respectively, we obtain

$$EF = AF \cdot \frac{\sin A}{\sin \angle AEF} \quad \text{and} \quad AF = AP \cdot \frac{\sin \angle APF}{\sin 90°}.$$

Hence $EF = AP \sin A$.

Construct a line through point F which is parallel to BC. Extend PD so as to meet the new line at H. Also, drop a perpendicular from E onto the line at G.



Notice that the circle with diameter PB passes through both F and D. Therefore, $\angle B + \angle FPD = 180°$. Consequently, $\angle FPH = \angle B$. Similarly, $\angle HPG = \angle C$. From part (i),

$$PA \sin A = EF \geq FH + HG.$$

Now,

$$FH = PF \sin \angle FPH = PF \sin B \quad \text{and} \quad HG = PE \sin \angle HPG = PE \sin C.$$

Therefore,

$$PA \geq PF \frac{\sin B}{\sin A} + PE \frac{\sin C}{\sin A}.$$

Similarly,

$$PB \geq PD \frac{\sin C}{\sin B} + PF \frac{\sin A}{\sin B} \quad \text{and} \quad PC \geq PE \frac{\sin A}{\sin C} + PD \frac{\sin B}{\sin C}.$$

Thus

$$PA + PB + PC \geq PF(\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B}) + PE(\frac{\sin C}{\sin A} + \frac{\sin A}{\sin C}) + PD(\frac{\sin C}{\sin B} + \frac{\sin B}{\sin C}).$$

Finally, $x + 1/x \geq 2$ for any $x > 0$. It follows that

$$\frac{\sin B}{\sin A} + \frac{\sin A}{\sin B} \geq 2.$$

A similar inequality holds for the other two terms. Hence

$$AP + BP + CP \geq 2(PE + PD + PF).$$

4. The hypothesis produces factorization formulas for a host of different numbers. The trick is to reduce the factors steadily until they are so small that few choices are left. Let us introduce the standard notation : if m and n are integers, we write $m|n$ to mean that m divides n. Multiplying out,

$$(ab-1)(bc-1)(ca-1) = a^2b^2c^2 - a^2bc - ab^2c - abc^2 + ab + bc + ca - 1. \qquad (12)$$

Since the above number is divisible by abc, and thus by c, we see that $c|(ab-1)$. Let k be a positive integer so that $ab - 1 = kc$. Since the number in equation (12) is divisible by abc, it follows that abc divides $ab + bc + ca - 1$. Hence

$$abc|c(b+a) + ab - 1 = c(b+a+k),$$

and so $ab|(b+a+k)$; this in turn implies that $b|(a+k)$. Let $a+k = pb$. Note that $a > kc/b > k$. Then

$$k < a < b \quad \Rightarrow \quad pb = a + k < 2b.$$

Therefore, $p = 1$. It follows that $a + k = b$. Hence $ab|(b + a + k) = 2b$. Thus $a = 2$. Since $k < a$, k must be 1. Then $b = a + k = 3$. Finally, $c = ab - 1 = 5$.

5. The function $f(x) = x^{10}$ is convex for $x \geq 0$. Therefore, if $\alpha_1, \alpha_2, \alpha_3$, and α_4 are nonnegative numbers which add up to 1, then

$$f(\alpha_1 a + \alpha_2 b + \alpha_3 c + \alpha_4 d) \leq \alpha_1 f(a) + \alpha_2 f(b) + \alpha_3 f(c) + \alpha_4 f(d).$$

Hence

$$\begin{aligned}
(0.1a + 0.2b + 0.3c + 0.4d)^{10} &\leq 0.1a^{10} + 0.2b^{10} + 0.3c^{10} + 0.4d^{10} \\
(0.4a + 0.3b + 0.2c + 0.1d)^{10} &\leq 0.4a^{10} + 0.3b^{10} + 0.2c^{10} + 0.1d^{10} \\
(0.2a + 0.4b + 0.1c + 0.3d)^{10} &\leq 0.2a^{10} + 0.4b^{10} + 0.1c^{10} + 0.3d^{10} \\
(0.3a + 0.1b + 0.4c + 0.2d)^{10} &\leq 0.3a^{10} + 0.1b^{10} + 0.4c^{10} + 0.2d^{10}.
\end{aligned}$$

The result follows by adding up the previous four lines.

Solution to Singapore Mathematical Olympiad 1996
Part A

1. Strategy: Analyze the hypotheses in the question carefully as that can simplify the calculations substantially.

   Observe that if a is a root, so is $-a$ since each term of the equation is an even function in x. Thus the roots of the equation must be of the form $-3r$, $-r$, r, $3r$.

   Hence we have

$$(x^2 - r^2)(x^2 - 9r^2) = x^4 - (3m-1)x^2 + (m-1)^2.$$

Comparing coefficients, we have

$$10r^2 = 3m - 1 \qquad \text{and} \qquad 9r^4 = (m-1)^2.$$

Eliminating r, we get $19m^2 - 146m + 91 = 0$. Hence sum of the two values of $m = \frac{146}{19} = 7\frac{13}{19}$, and the answer is (C).

2. Strategy: For geometric problems, it is often helpful to start with a diagram.

   Let O be the centre of the circle. Join OA, OB, OC, OD and AD.

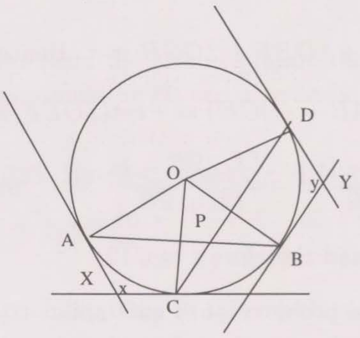


Fig. 1

To relate $\angle APD$, x and y, it is helpful first to relate these angles with angles subtended at the centre. First $\angle DOB = 180° - y$, $\angle AOC = 180° - x$. It follows that

$$\angle DAB = \frac{1}{2}\angle DOB = 90° - \frac{y}{2} \qquad \text{and} \qquad \angle ADC = \frac{1}{2}\angle AOC = 90° - \frac{x}{2}.$$

Therefore, $\angle APD = 180° - \angle DAB - \angle ADC = \frac{1}{2}(x+y)$.

3. Strategy: Special features can be very useful in solving a problem. In this problem, it is important to observe that the two terms on the left hand side of the equation are reciprocals of each other.

   Let $y = \sqrt{a + \sqrt{a^2-1}}$. Then $\sqrt{a - \sqrt{a^2-1}} = \dfrac{a^2 - (a^2-1)}{\sqrt{a+\sqrt{a^2-1}}} = \dfrac{1}{y}$. Thus the equation can be rewritten as

$$y^x + \frac{1}{y^x} = 2a \qquad \text{or} \qquad (y^x)^2 - 2ay^x + 1 = 0.$$

   Thus $y^x = \dfrac{2a \pm \sqrt{4a^2-4}}{2} = a \pm \sqrt{a^2-1} = y^2$ or y^{-2}. It follows that $x = 2, -2$.

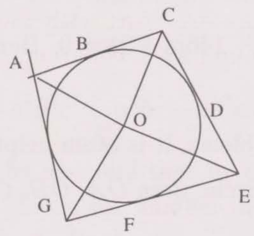
4. Strategy: In this problem, it is helpful to find some quantities which remain unchanged after each encounter.

   Suppose that after the first k encounters, there are x_k white, y_k grey and z_k black chameleons. It is easy to see that $(x_{k+1}, y_{k+1}, z_{k+1})$ must be either (x_k-1, y_k-1, z_k+2), (x_k-1, y_k+2, z_k-1), (x_k+2, y_k-1, z_k-1) or (x_k, y_k, z_k). Observe that in all cases, we must have $x_{k+1} - y_{k+1} \equiv x_k - y_k \bmod 3$. Initially we have $x_0 - y_0 = 18 - 16 = 2$. Thus $x_k - y_k = 2 \bmod 3$ for all k. A simple check shows that (D) is not possible, since $11 - 7 = 4 \equiv 1 \bmod 3$.

   Thus the answer is (D).

5. Strategy: Same as in Question 2.

   Let O be the centre and r be the radius of the circle.



   Then $\angle OAB + \angle OCD + \angle OEF + \angle OGH = \pi$. Hence

$$\cot(\angle OAB + \angle OCD) = -\cot(\angle OEF + \angle OGH), \text{ which implies}$$

$$\frac{\frac{AB}{r}\cdot\frac{CD}{r}-1}{\frac{AB}{r}+\frac{CD}{r}} = -\frac{\frac{EF}{r}\cdot\frac{GH}{r}-1}{\frac{EF}{r}+\frac{GH}{r}} \quad \text{or} \quad \frac{\frac{3}{r}\cdot\frac{4}{r}-1}{\frac{3}{r}+\frac{4}{r}} = -\frac{\frac{5}{r}\cdot\frac{6}{r}-1}{\frac{5}{r}+\frac{6}{r}}.$$

   This leads to $r^2 = 19$, and therefore $r = \sqrt{19}$.

6. Strategy: If the original problem looks unfamiliar, try to reformulate it into one which looks more familiar. This problem can be reformulated into a problem on geometric progression.

   Let $b_n = \ln a_n$. Then

$$a_n^{\ln a_n} = a_{n+1}^{\ln a_{n-1}} \Rightarrow (\ln a_n)^2 = (\ln a_{n-1})(\ln a_{n+1})$$
$$\Rightarrow b_n^2 = b_{n-1}b_{n+1} \Rightarrow \frac{b_n}{b_{n-1}} = \frac{b_{n+1}}{b_n}.$$

   Thus the sequence $b_1, b_2, \ldots, b_n, \ldots$ forms a geometric progression. Now $b_0 = \ln 2$, $b_1 = \ln 4 = 2\ln 2$. Thus the common ratio is $(2\ln 2)/\ln 2 = 2$, and we have $b_n = b_0 \cdot 2^n = 2^n \ln 2$.

   Therefore, $a_n = e^{b_n} = e^{2^n \ln 2} = 2^{2^n}$.

7. Strategy: Try to exhaust all possibilities.

   It is easy to construct polyhedra with 6, 8, 9, 10 edges respectively. Observe that if one of the faces of a polyhedron is not a triangle, then the polyhedron has at least 8 edges. Also observe that if each face of a polyhedron is a triangle, then the number of edges= $(3/2)\times$ number of faces, which must be a multiple of 3. The above two observations imply that there is no polyhedron with exactly 7 edges.

   Therefore the answer is (B).

8. Strategy: Specialize and generalize. When one sees a complicated problem, it is sometimes helpful to first look at a similar but simpler problem or some special cases. This may shed light on the complicated problem. In this problem, it is helpful to consider first the simpler problems when the summation involves only a few terms, say one, two or three terms.

   When one considers the problem when there are only two terms, one is easily led to the following observation which will be useful for the given problem: for fixed real numbers a, b with $a < b$, the function $f(x) = |x-a| + |x-b|$ attains its minimum value $b-a$ at any x satisfying $a \le x \le b$. By rewriting

$$\sum_{k=0}^{1996} |x - \sqrt{k}| = \sum_{k=0}^{997} \left(|x-\sqrt{k}| + |x - \sqrt{1996-k}|\right) + |x - \sqrt{998}|,$$

it is then easy to see that $\sum_{k=0}^{1996} |x - \sqrt{k}|$ attains its minimum when $x = \sqrt{998}$.

9. Strategy: Try to work backwards. Start with the equation, and try to see what restrictions are imposed on x, y, z, t assuming the equation.

   Observe that $7|4550$ and $7|70$, and that among the four given numbers, only 70 is divisible by 7. If $x = 70$, then it follows from the given equation that $7|yt$, which is not possible. If $y = 70$, then $7|xz$, which is also not possible. Thus either z or t must be 70.

   Observe that $13|4550$ and $13|325$. By a similar argument as before, one concludes that either z or t must be 325.

   If $t = 325$, $z = 70$, then it follows from the given equation that $x(y - z) = 4550 - 325y < 0$, which implies that $y < z$. Hence y must be 60 and x must be 185. It is easy to see that this is impossible.

   Thus we must have $t = 70$, $z = 325$. By direct checking, it follows easily that $x = 60$, $y = 185$.

   Therefore, $x = 60$, $y = 185$, $z = 325$, $t = 70$.

10. Strategy: Draw a three dimensional diagram. In this problem, it is also helpful to introduce a coordinate system.

    We choose a rectangular coordinate system so that the coordinates of A, B, C, D are $(1/\sqrt{3}, \pm 1/\sqrt{3}, \pm 1/\sqrt{3})$. Let the coordinates of P be (u, v, w) so that $u^2 + v^2 + w^2 = 1$. Then

$$\cos \angle POA = \frac{\vec{OP} \cdot \vec{OA}}{|\vec{OP}||\vec{OA}|} = \frac{u}{\sqrt{3}} + \frac{v}{\sqrt{3}} + \frac{w}{\sqrt{3}}.$$

    Similar calculations give

$$\begin{aligned} &\cos^2 \angle POA + \cos^2 \angle POB + \cos^2 \angle POC + \cos^2 \angle POD \\ &= \left(\tfrac{u}{\sqrt{3}} + \tfrac{v}{\sqrt{3}} + \tfrac{w}{\sqrt{3}}\right)^2 + \left(\tfrac{u}{\sqrt{3}} + \tfrac{v}{\sqrt{3}} - \tfrac{w}{\sqrt{3}}\right)^2 + \left(\tfrac{u}{\sqrt{3}} - \tfrac{v}{\sqrt{3}} + \tfrac{w}{\sqrt{3}}\right)^2 + \left(\tfrac{u}{\sqrt{3}} - \tfrac{v}{\sqrt{3}} - \tfrac{w}{\sqrt{3}}\right)^2 \\ &= \tfrac{4}{3}\left(u^2 + v^2 + w^2\right) = \tfrac{4}{3}. \end{aligned}$$

Part B

1. Strategy: In this problem, it is helpful to reformulate it into a simple geometrical problem of finding the volume of a certain polyhedron.

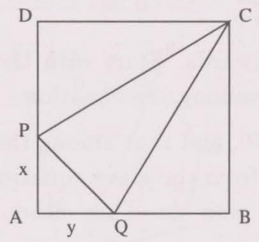
   Suppose that the three numbers drawn are x, y and z. Then (x, y, z) is a point in the cube $I^3 = \{ (x, y, z) : 0 \le x,\ y,\ z \le 1 \}$ in $\mathbf{R}^3$, and each point in I^3 arises from one such triple. The three numbers fail to form the lengths of the sides of a triangle if and only if

$$x \ge y + z \qquad \text{or} \quad y \ge x + z \qquad \text{or} \quad z \ge x + y.$$

   These three inequalities correspond to three disjoint regions in I^3, each being a tetrahedron of volume $\frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{6}$.

   Therefore the probability of forming a triangle $= 1 - 3 \times \frac{1}{6} = \frac{1}{2}$.

2. Let $|AP| = x$ and $|AQ| = y$.



Then

$$\tan \angle PCD = \frac{|DP|}{|CD|} = 1 - x \quad \text{and} \quad \tan \angle QCB = \frac{|QB|}{|BC|} = 1 - y.$$

Now

$$\begin{aligned}\tan \angle PCQ = \cot(90^\circ - \angle PCQ) &= \frac{1}{\tan(\angle PCD + \angle QCB)} \\ &= \frac{1 - \tan \angle PCD \tan \angle QCB}{\tan \angle PCD + \tan \angle QCB} \\ &= \frac{1 - (1-x)(1-y)}{(1-x)+(1-y)} = \frac{x + y - xy}{2 - x - y}.\end{aligned}$$

On the other hand,

$$\begin{aligned}|AP| + |AQ| + |PQ| = 2 \quad &\Rightarrow x + y + \sqrt{x^2+y^2} = 2 \\ &\Rightarrow (x+y)^2 = (2 - \sqrt{x^2+y^2})^2 \\ &\Rightarrow 2xy = 4 - 4\sqrt{x^2+y^2} = 4 - 4(2 - x - y) \\ &\Rightarrow xy = -2 + 2x + 2y.\end{aligned}$$

Hence $\tan \angle PCQ = \dfrac{x + y - (-2 + 2x + 2y)}{2 - x - y} = 1$. Clearly, $\angle PCQ = 45^\circ$.

3. Strategy: Proof by contradiction. First assume there is a solution to the equation, and try to see that this cannot happen. This involves getting contradictions in all the different possible cases.

If $x \equiv 1 \bmod 4$, then the given equation implies that $(-1)^n - 1 \equiv 1 + (-1)^n \bmod 4$ or equivalently, $-1 \equiv 1 \bmod 4$, which is impossible.

If $x \equiv -1 \bmod 4$, then the given equation implies that $1 - (-1)^n \equiv 1 + (-1)^n \bmod 4$ or equivalently, $2(-1)^n \equiv 0 \bmod 4$, which is again impossible.

Thus it remains to consider the case when x is even.

If x is even, then $(x+2)^n - x^n$ is divisible by 2^n. However, $1 + 7^n \equiv 2 \bmod 4$ if n is even; and $1 + 7^n \equiv 8 \bmod 16$ when n is odd. So 2^n divides $1 + 7^n$ only when n is 1 or 3. Clearly n cannot be 1.

When $n = 3$, we have

$$6x^2 + 12x + 8 = 1 + 343 \quad or \quad x^2 + 2x - 56 = 0.$$

Since the discriminant $= 2^2 - 4 \times 1 \times (-56) = 228$ which is not a perfect square, the equation has no integer solutions.

Therefore in all cases, the given equation has no integer solutions.

4. Strategy: Try to work backwards. It is also helpful to observe the special feature that the equation is symmetric in x, y, z.

   The given equation is symmetric in x, y, z. First we consider the case when $x \le y \le z$, and write the equation in the form

$$z = wx^2y^2 - \frac{x^3+y^3}{z^2}. \tag{13}$$

   Since x, y, z, w are integers, it follows that $(x^3+y^3)/z^2$ is also an integer, and thus

$$x^3 + y^3 \ge z^2. \tag{14}$$

   Moreover, it follows from (13) and the assumptions $x/z \le 1$, $y/z \le 1$ that $z \ge wx^2y^2 - (x+y) \ge -z$ which implies that

$$z^2 \ge [wx^2y^2 - (x+y)]^2. \tag{15}$$

   Now (14) and (15) imply that $w^2x^4y^4 < 2wx^2y^2(x+y) + x^3 + y^3$ which implies that

$$wxy < 2\Big(\frac{1}{x} + \frac{1}{y}\Big) + \frac{1}{wx^3} + \frac{1}{wy^3}. \tag{16}$$

   If $x \ge 2$, then the right hand side of (16) ≤ 3, while the left hand side of (16) ≥ 4. This contradiction shows that $x = 1$. By substituting $x = 1$ into (16), we get

$$wy < 2 + \frac{2}{y} + \frac{1}{w} + \frac{1}{wy^3}. \tag{17}$$

   If $y \ge 4$, then the right hand side of (17) < 4, while the left hand side of (17) ≥ 4, which is impossible. Thus $y \le 3$.

   By (13), we have

$$z^2 | (1 + y^3) \tag{18}$$

   If $y = 1$, then $z = 1$ and $w = 3$.

   If $y = 2$, then if follows from (18) that $z = 3$ and thus $w = 1$.

   Also it follows from (18) that y cannot be 3.

   Since the given equation is symmetric in x, y, z, it is easy to see that all the solutions are

$$\begin{aligned}(x,y,z,w) = \;& (1,1,1,3),\ (1,2,3,1),\ (1,3,2,1),\ (2,1,3,1),\ (2,3,1,1)\\ & (3,1,2,1),\ (3,2,1,1).\end{aligned}$$

Singapore Mathematical Olympiad 1997
Part A

1. Since $x^2 + Ax + B = 0$ has roots r and s, we have $r + s = -A$ and $rs = B$. Now the equation $x^2 + Cx + D = 0$ has repeated roots $r - s$, thus we have

$$D = (r-s)^2 = (r+s)^2 - 4rs = A^2 - 4B.$$

2. Extend AD and BC to meet at a point E. Then $AE = AB/\cos 60^\circ = 4/0.5 = 8$. Thus $DE = 8 - 5 = 3$. $\angle ECD = 180^\circ - \angle DCB = 180^\circ - 120^\circ = 60^\circ$. Then $EC = DE/\sin 60^\circ = 3/(\sqrt{3}/2) = 2\sqrt{3}$. $BE = AE \sin 60^\circ = 8 \times (\sqrt{3}/2) = 4\sqrt{3}$. Hence $BC = 4\sqrt{3} - 2\sqrt{3} = 2\sqrt{3}$. Also $DC = DE/\tan \angle ECD = 3/\tan 60^\circ = \sqrt{3}$. Thus $BC/CD = 2\sqrt{3}/\sqrt{3} = 2$.

3. Strategy: Try to exhaust all the possible cases.

Let x_1, x_2, x_3, x_4 be the four chosen numbers. First we consider the case when $x_1x_2x_3x_4$ ends with the number 9. This happens precisely when $x_1x_2x_3$ ends with 1, 3, 7 or 9, while x_4 ends with 9, 3, 7, 1 respectively. Observe also that $x_1x_2x_3$ ends with 1, 3, 7 or 9 precisely when all x_1, x_2 and x_3 end with 1, 3, 7 or 9. Thus the probability that $x_1x_2x_3x_4$ ends with 9 is $\frac{4}{10} \times \frac{4}{10} \times \frac{4}{10} \times \frac{1}{10} = \frac{4}{625}$. Similarly, the probability that $x_1x_2x_3x_4$ ends with 1 is also $\frac{4}{625}$. Thus probability that $x_1x_2x_3x_4$ ends with 1 or 9 is $\frac{4}{625} + \frac{4}{625} = \frac{8}{625}$.

4. Strategy: Exhaust all the cases according to the signs of x, y. Drawing a diagram showing the boundaries of the region in each case also helps.

First we consider the first quadrant where $x, y \geq 0$. The inequality becomes $x+y+(x+y) \leq 2$ or $x + y \leq 1$. Thus the area of the region in the first quadrant is $\frac{1}{2}$. Next we consider the second quadrant where $x \leq 0$, $y \geq 0$. If $x+y \geq 0$, the inequality becomes $-x+y+x+y \leq 2$ or $y \leq 1$. This corresponds to the triangle $0 \leq y \leq 1$, $0 \leq -x \leq y$ with area $\frac{1}{2}$. If $x + y < 0$, the inequality becomes $-x + y - (x + y) \leq 2$ or simply $-x \leq 1$. This corresponds to the triangle $0 \leq -x \leq 1$, $0 \leq y \leq -x$ with area $\frac{1}{2}$. Since the region is symmetric with respect to the x-axis,

$$\text{total area } = 2\left(\frac{1}{2} + \frac{1}{2} + \frac{1}{2}\right) = 3.$$

5. For any integer $k \geq 1$, we let

$$\binom{x}{k} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!}.$$

Now we consider the polynomial

$$g(x) = 1 + \binom{x}{1} + \binom{x}{2} + \binom{x}{3} + \binom{x}{4} + \binom{x}{5}.$$

Using the expression $2^n = \sum_{r=0}^{n} \binom{n}{r}$, it is easy to check that $g(k) = 2^k$ for $k = 0, 1, 2, 3, 4, 5$. Since both $f(x)$ and $g(x)$ are polynomials of degree 5 and they agree at 6 distinct points, we must have $f(x) \equiv g(x)$. Thus

$$f(6) = g(6) = 1 + \binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} = 2^6 - 1$$

6. Strategy: Try to reformulate an unfamiliar problem into a familiar one. It turns out that the bizarre equation in this problem is just a quadratic equation in disguise!

Let $\log(x+2) = m + \alpha$ and $\log x = n + \beta$, where $m, n \in \mathbf{N}$ and $0 \leq \alpha, \beta < 1$. Then the given equation becomes $\alpha + \beta = 1$. Thus we have

$$\log(x+2) + \log x = m + n + 1 \in \mathbf{N}.$$

Write $k = m + n + 1 \in \mathbf{N}$. Thus we have

$$x(x+2) = 10^k \text{ or } x^2 + 2x - 10^k = 0.$$

This implies

$$x = \frac{-2 \pm \sqrt{4 + 4 \cdot 10^k}}{2} = \pm\sqrt{10^k + 1} - 1.$$

The smallest solution $x > 5$ is thus given by $\sqrt{101} - 1$ when $k = 2$.

7. Since $AB = A0$ and M is the midpoint of BO, AM must be perpendicular to BO. Thus $\angle AOM = 45^\circ$. Then $AM = AO\cos 45^\circ = \sqrt{2}$ cm. Similarly, $AN = \sqrt{2}$ cm. Since M, N are the midpoints of BO and CO respectively, we have $MN = (1/2)BC = \sqrt{2}$ cm. Thus, $\triangle AMN$ is an equilateral triangle with all sides of length $\sqrt{2}$ cm. Thus area of $\triangle AMN = \frac{1}{2}(\sqrt{2})^2 \sin 60^\circ = \frac{\sqrt{3}}{2}$ cm^2.

8. From the graph of $y = ax^2 + bx + c$, we know that as $x \to \infty$, $y \to -\infty$. Thus $a < 0$. Also, $c = y(0) < 0$. Thus, $ac > 0$. The x-coordinate of the vertex of the parabola is $-b/2a$. Hence we have $0 < -b/2a < 1$. Thus $b > 0$ and $2a + b < 0$. Since $a < 0$ and $b > 0$, we have $ab < 0$ and $2a - b < 0$. When $x = -1$, we have $y(-1) = a - b + c < 0$. When $x = 1$, we have $y(1) = a + b + c > 0$. In summary, there are 2 expressions ac and $a + b + c$ which are always positive.

9. Since $\triangle GFD \sim \triangle GBC$ and $\triangle ABE \sim \triangle CGE$, we have

$$\frac{GF}{GB} = \frac{GD}{GC} \quad \text{and} \quad \frac{AB}{CG} = \frac{BE}{GE}.$$

Then

$$\begin{aligned} BF = BG + GF &= BG\left(1 + \frac{GD}{GC}\right) = BG\left(\frac{GC + GD}{GC}\right) \\ &= BG\left(\frac{AB}{CG}\right) = BG\left(\frac{BE}{GE}\right) \\ &= BG\left(\frac{BE}{BG - BE}\right). \end{aligned}$$

Since $BG = 60$ and $BF = 90$, we have $90 = 60 \times \frac{BE}{60 - BE}$ and thus $BE = 36$.

10. Strategy: Try to exploit the special feature that all factors of 3^y are powers of 3.

There are 2 cases:

Case 1. $x - 5 > 0$ and $x - 77 > 0$. Since all the factors of 3^y are powers of 3, we have $x - 5 = 3^k$ and $x - 77 = 3^{y-k}$ for some integer k such that $k \geq y - k \geq 0$. Then $(x-5) - (x-77) = 3^k - 3^{y-k}$. Thus $3^{y-k}(3^{2k-y} - 1) = 72 = 2^3 3^2$. Since 2 does not divide 3^{y-k} and 3 does not divide $3^{2k-y} - 1$, we must have $3^{y-k} = 3^2$ and $3^{2k-y} - 1 = 2^3$. Thus, $y - k = 2$ and $2k - y = 2$. Solving the two equations, we get $k = 4$, $y = 6$. Then $x = 5 + 3^4 = 86$. A simple check shows that $(x_1, y_1) = (86, 6)$ satisfies the given equation.

Case 2. $x - 5 < 0$ and $x - 77 < 0$. Similar to case 1, we must have $x - 5 = -3^k$ and $x - 77 = -3^{y-k}$ for some integer k such that $y - k \geq k \geq 0$. Then $(x-5)(x-77) = -3^k + 3^{y-k}$. Thus $3^k(3^{y-2k} - 1) = 72 = 2^3 3^2$. This means that $3^k = 3^2$ and $3^{y-2k} - 1 = 2^3$. Thus, $k = 2$, $y = 6$, and $x = 5 - 3^2 = -4$. A simple check shows that $(x_2, y_2) = (-4, 6)$ satisfies the given equation.

Hence $x_1 + x_2 = 86 - 4 = 82$.

Part B

1. Area of $\triangle MAN +$ area of $\triangle NBL +$ area of $\triangle LCM <$ area of $\triangle ABC$. Hence,

$$\frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot MA \cdot AN + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot NB \cdot BL + \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot LC \cdot CM < \frac{1}{2} \cdot \frac{\sqrt{3}}{2} \cdot BC^2.$$

Therefore, $MA \cdot AN + NB \cdot BL + LC \cdot CM < BC^2$.

2. Strategy: Specialize and generalize. A direct checking with the first few natural numbers will suggest that such numbers are of the form $p-1$, where p is some prime number.

First we show that a natural number n indeed has the desired property if and only if $n = p-1$ for some prime number p. Since $\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$ is an integer, it follows that if $n+1$ is prime, then $\binom{n}{k}$ is divisible by $k+1$. Conversely, assume that $n+1$ is a composite number. Let q be that smallest prime number that divides $n+1$. Then we must have $2 \le q \le n/2$, and thus $1 \le q-1 \le n-1$. Since q divides $n+1$, q cannot divide $n, n-1, \ldots, n-q+2$. Thus q cannot divide $\binom{n}{q-1}$. Now the prime numbers between 51 and 91 are 53, 59, 61, 67, 71, 73, 79, 83 and 89. Thus, $n = 52, 58, 60, 66, 70, 72, 78, 82, 88$.

3. Strategy: Try to exhaust all possible cases.

Since N has exactly 6 positive factors, N must be of the form $N = p^5$ or $N = pq^2$ for some prime numbers p and q.

Case 1. $N = p^5$. Then d_1, d_2, d_3, d_4 are p, p^2, p^3, p^4 up to permutations. By (ii), we have $1 + p^5 = 5(p + p^2 + p^3 + p^4)$. This is not possible since $1 + p^5$ is not divisible by p.

Case 2. $N = pq^2$. Then d_1, d_2, d_3, d_4 are p, q, pq, q^2 up to permutations. By (ii), we have $1 + pq^2 = 5(p + q + pq + q^2)$. Upon rewriting, we get

$$p = \frac{5q^2 + 5q - 1}{q^2 - 5q - 5} = \frac{30q + 24}{q^2 - 5q - 5} + 5.$$

Since p is an integer, we must have $q^2 - 5q - 5 \le 30q + 24$, which yields $0 \le q \le 35$. A quick check shows that $q = 7$ is the only prime that leads to a prime value for p, namely $p = 31$. Thus $N = pq^2 = 31 \cdot 7^2 = 1519$.

4. Strategy: Try to study first the special case when $n = 3$ (the smallest n when the problem is non-trivial). It turns out that the proof for the general case is quite similar and not much more complicated.

Let $A_k = a_1 a_2 \cdots a_{k-1} a_{k+1} \cdots a_n$ for $1 \le k \le n$. Without loss of generality, we may assume that $a_1 \le a_2 \le \cdots \le a_n$ so that $A_1 \ge A_2 \ge \cdots \ge A_n$. We are going to maximize $\sum_{i=1}^{n} b_i A_i$ subject to the given conditions. It is obvious that we may assume $b_1 \ge b_2 \ge \cdots \ge b_n$ since $b_i \ge 0$, $\sum_{i=1}^{n} b_i = 1$ and $A_1 \ge A_2 \ge \cdots \ge A_n$. Since $0 \le b_i \le \frac{n-1}{n}$ and $A_1 \ge A_2 \ge \cdots \ge A_n$,

$$\begin{aligned} \textstyle\sum_{i=1}^{n} b_i A_i &\le \tfrac{n-1}{n} A_1 + \left(1 - \tfrac{n-1}{n}\right) A_2 \le a_3 a_4 \cdots a_n \left(\tfrac{n-1}{n} a_2 + \tfrac{1}{n} a_1\right) \\ &\le a_3 a_4 \cdots a_n \cdot (a_1 + a_2) \cdot \tfrac{n-1}{n} \qquad (\text{since } n \ge 2). \end{aligned}$$

Using $G.M. \le A.M.$, we have

$$\begin{aligned} a_3 a_4 \cdots a_n (a_1 + a_2) &\le \left(\frac{a_3 + a_4 + \cdots + a_n + (a_1 + a_2)}{n-1}\right)^{n-1} \\ &= \frac{1}{(n-1)^{n-1}} \qquad (\text{since } \sum_{i=1}^{n} a_i = 1). \end{aligned}$$

Hence,

$$\sum_{i=1}^{n} b_i A_i \le \frac{1}{(n-1)^{n-1}} \cdot \frac{n-1}{n} = \frac{1}{n(n-1)^{n-2}}.$$