Solutions to National Team Selection Tests

Prepared by Tay Tiong Seng and Wong Yan Loi

1994/95

1.1 It can be proved by induction that $f(n)$ is the number of ones in the binary representation of n.

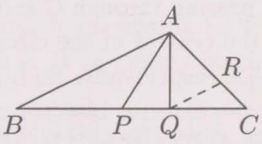
(i) There can be at most 10 ones in the binary representation of a natural number if it is less than or equal to $1994 = 11111001010_{(2)}$. Hence $M = 10$.

(ii) For any natural number n less than or equal to 1994, $f(n) = 10$ if and only if n is

$$\begin{aligned} 1023 &= 1111111111_{(2)}, \\ 1535 &= 10111111111_{(2)}, \\ 1791 &= 11011111111_{(2)}, \\ 1919 &= 11101111111_{(2)}, \\ 1983 &= 11110111111_{(2)}. \end{aligned}$$

1.2. **Stewart's theorem.** In $\triangle ABC$, D is a point on BC such that AD bisects $\angle A$. Then $AB : BD = AC : CD$.

1st solution



Applying Stewart's theorem to $\triangle ABQ$, we have $\dfrac{AB}{AQ} = \dfrac{BP}{PQ}$.

Given $BP \cdot CQ = BC \cdot PQ$, it follows that $\dfrac{BC}{CQ} = \dfrac{AB}{AQ}$.

Now let R be the point on AC such that QR is parallel to BA.

Then $\dfrac{AB}{RQ} = \dfrac{BC}{CQ} = \dfrac{AB}{AQ}$.

Hence $RQ = AQ$ and $\angle QAR = \angle QRA$.

Therefore $\angle PAC = \angle PAQ + \angle QAR = \frac{1}{2}(\angle BAQ + \angle QAR + \angle QRA) = \frac{\pi}{2}$.

2nd solution

Since $\dfrac{CB}{CQ} = \dfrac{PB}{PQ} = \dfrac{AB}{AQ}$, by Stewart's theorem, AC is the external angle bisector of $\angle BAQ$. Hence $\angle PAC = \dfrac{\pi}{2}$.

1.3. (i) Note that twice the total number of clappings is equal to $\sum_{x \in S} f(x)$ which cannot be the odd number $2 + 3 + 4 + \cdots + 1995$.

(ii) Let $n \geq 2$. For a group S_n of $4n - 2$ students, the following configuration gives an example in which $\{f(x) \mid x \in S_n\} = \{2, 4, 5, ..., 4n\}$.

where $G(x)$ is a polynomial with integer coefficients. Thus

$$k \geq |F(c) - F(0)| = c(k+1-c)|G(c)|, \quad \text{for each } c \in \{1, 2 \ldots, k\}. \tag{2}$$

The inequality $c(k+1-c) > k$ holds for each $c \in \{1, 2, \ldots, k-1\}$ which is not an empty set if $k \geq 3$. Thus for any c in this set, $|G(c)| < 1$. Since $G(c)$ is an integer, $G(c) = 0$. Thus $2, 3, \ldots, k-1$ are roots of $G(x)$, which yields

$$F(x) - F(0) = x(x-2)(x-3)\cdots(x-k+1)(x-k-1)H(x). \tag{3}$$

We still need to prove that $H(1) = H(k) = 0$. For both $c = 1$ and $c = k$, (3) implies that

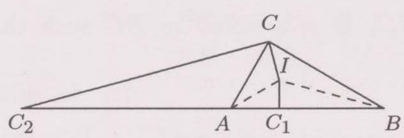
$$k \geq |F(c) - F(0)| = (k-2)! \cdot k \cdot |H(c)|.$$

Now $(k-2)! > 1$ since $k \geq 4$. Therefore $|H(c)| < 1$ and hence $H(c) = 0$.

For $k = 1, 2, 3$ we have the following counterexamples:

$$\begin{array}{ll} F(x) = x(2-x) & \text{for } k = 1, \\ F(x) = x(3-x) & \text{for } k = 2, \\ F(x) = x(4-x)(x-2)^2 & \text{for } k = 3. \end{array}$$

2.1. **1st solution**



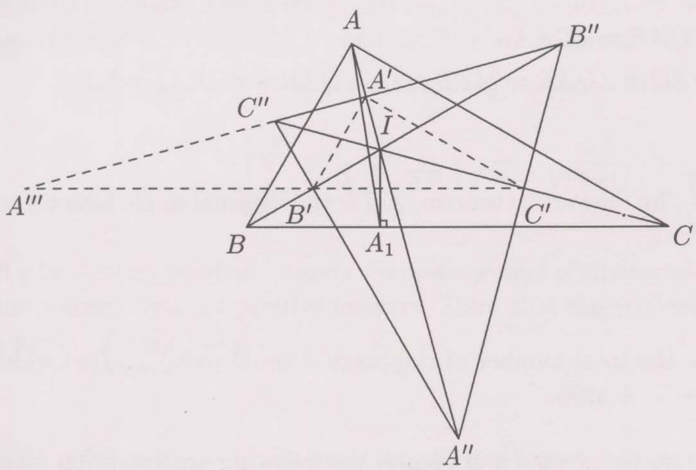
Let the line perpendicular to CI and passing through C meet AB at C_2. By analogy, we denote the points A_2 and B_2. It's obvious that the centres of the circumcircles of AIA_1, BIB_1 and CIC_1 are the middle points of A_2I, B_2I and C_2I, respectively. So it's sufficient to prove that A_2, B_2 and C_2 are collinear. Let's note that CC_2 is the exterior bisector of $\angle ACB$, and so $C_2A/C_2B = CA/CB$. By analogy $B_2A/B_2C = BA/BC$ and $A_2B/A_2C = AB/AC$. Thus

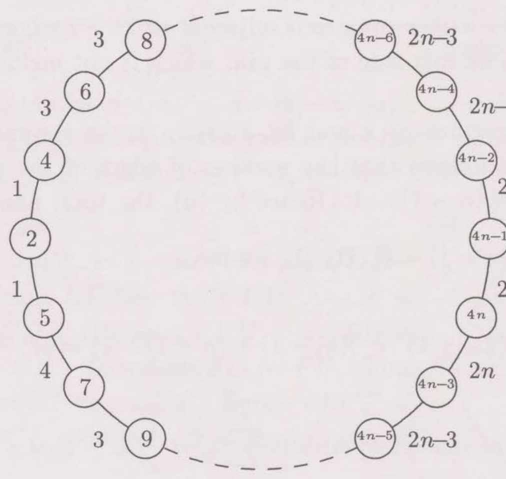
$$\frac{C_2A}{C_2B}\frac{B_2C}{B_2A}\frac{A_2B}{A_2C} = \frac{CA}{CB}\frac{BC}{BA}\frac{AB}{AC} = 1$$

and by Menelaus' Theorem[3], the points A_2, B_2 and C_2 are collinear.

2nd solution

Let A', B', C' be the midpoints of AI, BI, CI, respectively. Let the perpendicular bisectors of AI and BI meet at C''. A'' and B'' are similarly defined.





Each circle in the diagram represents a student x and the number in the circle represents $f(x)$. The number on each edge represents the number of times the two adjacent students clap hands with each other. Taking $n = 499$ gives an example of the problem.

2.1. (i) Denote the function $f(x)$ composed with itself n times by $f^{(n)}(x)$. Also let $g_0(x)$ be the identity function. Note that $f^{(2)}(x)$ is strictly increasing for $x > 0$. We shall prove by induction on n that $g_n(x)$ is strictly increasing for $x > 0$. It can easily be checked that $g_1(x)$ is strictly increasing for $x > 0$.

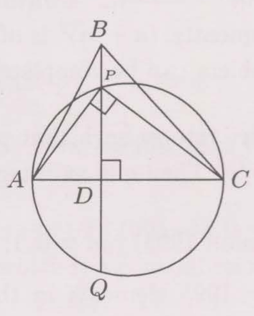
Suppose for $n \geq 2$, $g_1(x), ..., g_{n-1}(x)$ are strictly increasing. Let $x > y > 0$. We have

$$\begin{aligned} g_n(x) - g_n(y) &= (x-y) + (f(x) - f(y)) + (f^{(2)}(x) - f^{(2)}(y)) + \cdots + (f^{(n)}(x) - f^{(n)}(y)) \\ &= (g_1(x) - g_1(y)) + (g_{n-2}(f^{(2)}(x)) - g_{n-2}(f^{(2)}(y))) \ > \ 0. \end{aligned}$$

By induction, $g_n(x)$ is strictly increasing.

(ii) Note that $\dfrac{F_1}{F_2} = 1$ and $f(\dfrac{F_i}{F_{i+1}}) = \dfrac{F_{i+1}}{F_{i+2}}$. Hence $\dfrac{F_1}{F_2} + \cdots + \dfrac{F_{n+1}}{F_{n+2}} = g_n(1)$.

2.2.



Since $\triangle ADP$ is similar to $\triangle APC$, we have $AP/AD = AC/AP$.

Hence $AP^2 = AD \cdot AC = (BD \cot A) \cdot AC = 2(ABC) \cot A$, where (ABC) is the area of $\triangle ABC$.

Similarly, $AM^2 = 2(ABC) \cot A$.

Hence $AP = AQ = AM = AN = \sqrt{2(ABC) \cot A}$.

This shows that P, Q, M, N lie on the circle centered at A with radius $\sqrt{2(ABC) \cot A}$.

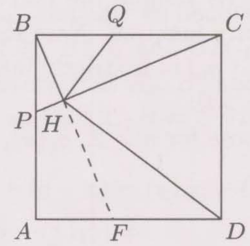
2.3. Let such a path be given. First the following facts are observed.
(i) The number of edges of the path is $nm - 1$.
(ii) By induction, each region with s squares is adjacent to $2s + 1$ edges of the path.
(iii) Each edge on the north or east side of the grid which is not included in the path corresponds to exactly one shaded region.
Let the number of shaded regions be k and let $s_1, s_2, ..., s_k$ be the number of squares in each of these regions. From (iii), it follows that the number of edges of the path on the north and east side of the grid is $(m-1) + (n-1) - k$. Hence by (ii), the total number of edges of the path is $\sum_{i=1}^{k}(2s_i + 1) \; + \; [(m-1) + (n-1) - k]$. By (i), we have

$$\sum_{i=1}^{k}(2s_i + 1) \; + \; [(m-1) + (n-1) - k] = nm - 1.$$

From this the total number of shaded squares is $\sum_{i=1}^{k} s_i = \frac{1}{2}(m-1)(n-1)$.

This problem appears in the American Mathematical Monthly. (See The American Mathematical Monthly, Vol.104, No.6, June-July 1997, p572-573.)

1995/96



1.1. Let BH intersect AD at F. Then $\triangle AFB$ is congruent to $\triangle BPC$. Hence $AF = BP = BQ$. Therefore $FD = QC$ and $QCDF$ is a rectangle. Since $\angle CHF = 90^o$, the circumcircle of the rectangle $QCDF$ passes through H. As QD is also a diameter of this circle, we have $\angle QHD = 90^o$.

1.2. Suppose there is a perfect square a^2 of the form $n2^k - 7$ for some positive integer n. Then a is necessarily odd. We shall show how to produce a perfect square of the form $n'2^{k+1} - 7$ for some positive integer n'. If n is even, then $a^2 = (n/2)2^{k+1} - 7$ is of the required form. Suppose that n is odd. We wish to choose a positive integer m such that $(a+m)^2$ is of the desired form.
Consider $(a+m)^2 = a^2 + 2am + m^2 = -7 + n2^k + m(m+2a)$. If we choose $m = 2^{k-1}$, then $m(m+2a)$ is an odd multiple of 2^k. Consequently, $(a+m)^2$ is of the form $n'2^{k+1} - 7$ for some positive integer n'. Now the solution of this problem can be completed by induction on k.

1.3. Let p be the smallest positive integer such that $pa \equiv 0 \pmod{1995}$, i.e. $pa = 1995k$ for some positive integer k. Let $q = 1995/p$. Then q is an integer and it divides a. We claim that

$$S = \{ma + nb \pmod{1995} \mid m = 0, 1, \ldots, \; p-1, n = 0, 1, \ldots, q-1\}$$

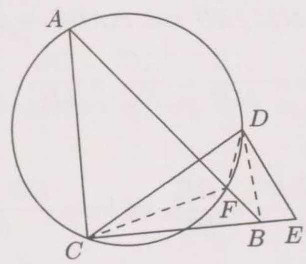
First note that there are $pq = 1995$ elements in the set on the right hand side. It suffices to prove that the elements are distinct. Suppose that $ma + nb \equiv m'a + n'b \pmod{1995}$. Then $(m - m')a + (n - n')b = 1995\ell$ for some integer ℓ. Since q divides 1995 and a, and q is relatively prime to b, we have q divides $(n - n')$. But $|n - n'| \leq q - 1$, so $n - n' = 0$. Consequently, $m = m'$. This completes the proof of the claim.

Consider the following sequence:

$$\underbrace{a, a, \ldots, a, b}_{p \text{ terms}}, \underbrace{-a, -a, \ldots, -a, b}_{p \text{ terms}}, \underbrace{a, a, \ldots, a, b}_{p \text{ terms}}, \ldots, \underbrace{(-1)^q a, (-1)^q a, \ldots, (-1)^q a, b}_{p \text{ terms}}$$

In this sequence, there are q blocks of $a, a, \ldots, a, b$ or $-a, -a, \ldots, -a, b$ making a total of $pq = 1995$ terms. For each $i = 1, 2 \ldots, 1995$, let s_i be the sum of the first i terms of this sequence. Then by the result above, $\{s_1, s_2 \ldots, s_{1995}\} = S$ and $s_{i+1} - s_i = \pm a$ or $\pm b \pmod{1995}$.

2.1. Since $\angle CDF = \angle CAF = 45^o$, we have $\angle FDE = \angle CDE - \angle CDF = 45^o = \angle CDF$. Hence DF bisects $\angle CDE$. As $CB = CD$, we have $\angle CBD = \angle CDB$. Hence $\angle FBD = \angle CBD - 45^o = \angle CDB - 45^o = \angle FDB$. Therefore $FD = FB$. This shows that $\triangle BCF$ is congruent to $\triangle DCF$. Hence $\angle BCF = \angle DCF$ and CF bisects $\angle DCE$. Therefore F is the incentre of $\triangle CDE$.



2.2. Let $\mathbb{N}$ be the set of all natural numbers. Let $A = \{n^2 \mid n \in \mathbb{N}\}$. Let $\mathbb{N} \setminus A = \{n_1, n_2, n_3, \ldots\}$. Define f as follows:

$$f(n) = \begin{cases} 1 & \text{if } n = 1, \\ n_{2i} & \text{if } n = n_{2i-1}, \quad i = 1, 2, \ldots \\ n_{2i-1}^2 & \text{if } n = n_{2i}, \quad i = 1, 2, \ldots \\ n_{2i}^{2^k} & \text{if } n = n_{2i-1}^{2^k}, \quad k = 1, 2, \ldots \\ n_{2i-1}^{2^{k+1}} & \text{if } n = n_{2i}^{2^k}, \quad k = 1, 2, \ldots \end{cases}$$

Then $f : \mathbb{N} \longrightarrow \mathbb{N}$ satisfies the requirement $f(f(n)) = n^2$.

(Note: The function above comes from the following consideration. First, $f(1)$ must be 1. Let $f(2) = 3$. Then $f(3) = 2^2$, $f(2^2) = 3^2$, $f(3^2) = 2^4$ etc.. Next, let $f(5) = 6$. Continuing as before, we have $f(6) = 5^2$, $f(5^2) = 6^2$, $f(6^2) = 5^4$ etc..)

2.3. Let $N = \{1, 2, ..., 1995\}$. Let q be an integer with $1 \leq q \leq m$. We shall prove the following statement *S(q)* by induction (on q):

S(q): There exists a subset I_q of N such that $\sum_{i \in I_q} n_i = q$.

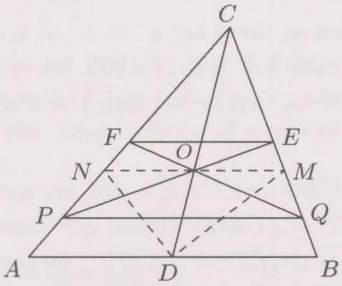
$S(1)$ is true because one of the n_i's must be 1. Now assume that for some q with $1 \leq q < m$, $S(i)$ is true for $i \leq q$. Then $|I_q| \leq q$ and 1994.

If $n_i > q+1$ for all $i \in N \setminus I_q$, then $\sum_{i \in N} n_i \geq q + (q+2)(1995 - |I_q|) = (1996 - |I_q|)q + 2(1995 - |I_q|) \geq 2q + 2(1995 - |I_q|) \geq 2q + 2(1995 - q) = 3990$, which is a contradiction. Hence, there exists $j \in N \setminus I_q$ such that $n_j \leq q + 1$. Let $a = \min\{n_i : i \notin I_q\}$. Then $a \leq q + 1$ and $a - 1 \leq q$. Thus $S(a-1)$ is true. By the choice of a, there exists $J \subseteq I_q$ such that $a - 1 = \sum_{i \in J} n_i$. Therefore, $q + 1 = q + a - (a-1) = \sum_{i \in I_q \setminus J} n_i + a$. Thus, $S(q+1)$ is true.

This problem appears in the American Mathematical Monthly with 1995 replaced by k and 3990 replaced by $2k$. The proof above works for the general case too. See (The American Mathematical Monthly, Vol.105, No.3, March 1998, pg 273-274.)

1996/97

1.1. Since DM and DN are angle bisectors of $\angle BDC$ and $\angle ADC$ respectively, by Stewart's theorem, we have



$$\frac{BM}{MC} = \frac{DB}{DC} \quad \text{and} \quad \frac{AN}{NC} = \frac{AD}{DC}.$$

As $AD = DB$, we have $\frac{BM}{MC} = \frac{AN}{NC}$.

Hence $NM /\!/ AB$ and $\triangle ABC \sim \triangle NMC$.

Therefore $\frac{AB}{NM} = \frac{AC}{NC} = \frac{BC}{MC}$.

Since $\frac{BM}{MC} = \frac{DB}{DC}$, we have $\frac{DB+DC}{DC} = \frac{BM+MC}{MC} = \frac{BC}{MC} = \frac{AB}{NM}$.

On the other hand, $FE = \frac{1}{2}AB = DB$. Therefore, $\frac{FE+DC}{DC} = \frac{2FE}{NM}$.

Consequently, $\frac{1}{FE} + \frac{1}{DC} = \frac{2}{NM}$.

Applying Menelaus's Theorem to $\triangle CMN$ for the lines EP and FQ and using the fact that $OM = ON$, we have

$$\frac{CP}{PN} = \frac{OM}{ON} \cdot \frac{CE}{ME} = \frac{CE}{ME} \quad \text{and} \quad \frac{CQ}{QM} = \frac{ON}{OM} \cdot \frac{FC}{FN} = \frac{FC}{FN}.$$

Since $FE /\!/ AB /\!/ NM$, we have $\frac{CE}{ME} = \frac{FC}{FN}$. Therefore $\frac{CQ}{QM} = \frac{CP}{PN}$ so that $FE /\!/ PQ$.

Hence $PQEF$ is a trapezoid and O is the intersection point of its two diagonals.

From this, it follows that $\frac{1}{FE} + \frac{1}{PQ} = \frac{2}{NM}$. Consequently, $PQ = DC$.

1.2. It can be shown that a_n satisfies the recurrence relation: $a_n = 2a_{n-1} + 2a_{n-2}$ with $a_1 = 3$ and $a_2 = 8$. Solving this difference equation gives

$$a_n = \left(\frac{1}{2} + \frac{1}{\sqrt{3}}\right)(1+\sqrt{3})^n + (-1)^{n+1}\left(\frac{1}{\sqrt{3}} - \frac{1}{2}\right)(\sqrt{3}-1)^n.$$

Next we shall show that $(\frac{1}{\sqrt{3}} - \frac{1}{2})(\sqrt{3}-1)^n < 0.5$ for $n \geq 1$. This is because

for $n \geq 1$, $0 < (\frac{1}{\sqrt{3}} - \frac{1}{2})(\sqrt{3}-1)^n \leq (\frac{1}{\sqrt{3}} - \frac{1}{2})(\sqrt{3}-1) < (1 - \frac{1}{2})(2-1) = 0.5$.

Thus $a_n = (\frac{1}{2} + \frac{1}{\sqrt{3}})(1+\sqrt{3})^n$ rounded off to the nearest integer.

1.3. **1st solution**

Let $x \in \mathbb{R}$. By letting $x = y + f(0)$, we obtain

$$f(f(x)) = f(f(y + f(0))) = f(0 + f(y)) = y + f(0) = x.$$

Hence for any $t_1, t_2 \in \mathbb{R}$, $f(t_1 + t_2) = f(t_1 + f(f(t_2))) = f(t_1) + f(t_2)$.
Next, consider any positive integer m such that $m \neq -f(x)$. We have

$$\frac{f(m + f(x))}{m + f(x)} = \frac{x + f(m)}{m + f(x)} = \frac{x + mf(1)}{m + f(x)}.$$

Since the set $\{\frac{f(t)}{t} \mid t \neq 0\}$ is finite, there exist distinct positive integers m_1, m_2 with $m_1, m_2 \neq -f(x)$ such that

$$\frac{f(m_1 + f(x))}{m_1 + f(x)} = \frac{f(m_2 + f(x))}{m_2 + f(x)}.$$

Hence $\frac{x + m_1 f(1)}{m_1 + f(x)} = \frac{x + m_2 f(1)}{m_2 + f(x)}$. From this, we have $f(x)f(1) = x$.

By letting $x = 1$, we obtain $[f(1)]^2 = 1$ so that $f(1) = \pm 1$. Consequently, $f(x) = \pm x$. Also the functions $f(x) = x$ and $f(x) = -x$ clearly satisfy the two given conditions.

2nd solution

(i) First we prove that $f(0) = 0$. Putting $x = 0 = y$, we have $f(f(0)) = f(0)$. If $f(0) = a$, then $f(0) = f(f(0)) = f(a)$. Thus $a + f(0) = f(0 + f(a)) = f(f(0)) = f(0)$, whence $a = 0$.
(ii) Putting $x = 0$, we have $f(f(y) = y$ for all y.
(iii) We will prove that $f(x) = \pm x$ for all x.
Suppose for some p, $f(p) = cp$ for some constant $c \neq \pm 1$. Then $f(p + f(p)) = p + f(p)$. Let $q = p + f(p)$. Then $q \neq 0$ and $f(q) = q$. Thus $f(q + f(q)) = q + f(q)$ and $f(2q) = 2q$. Inductively we have $f(nq) = nq$ for any positive integer n. Now $f(nq + f(p)) = p + f(nq)$. So $f(nq + cp) = p + nq$. Thus $f(nq + cp)/(nq + cp) = 1 - (c - 1)p/(nq + cp)$. Since $c - 1 \neq 0$ and there are infinitely many choices for n so that $nq + cp \neq 0$, this gives an infinite number of members in the set $\{f(x)/x\}$ contradicting the second condition. Thus $c = \pm 1$.
(iv) For $f(p) = p$, we will prove that $f(x) = x$ for all x.
If f$(-p) = p$, then $-p = f(f(-p)) = f(p) = p$ which is impossible. Thus $f(-p) = -p$.
Suppose there exists r such that $f(r) = -r$. Then $f(r + f(p)) = p + f(r)$, i.e., $f(r + p) = p - r$. Therefore $f(r + p)/(r + p) = (p - r)/(r + p) \neq \pm 1$. (Note that the denominator is not zero.)
(v) From the above we conclude that either $f(x) = x$ for all x or $f(x) = -x$ for all x.
Clearly these functions satisfy the two given conditions. Thus these are the only two functions required.

2.1. Let a, b, c, d represent the numbers at any stage subsequent to the initial one. Then $a+b+c+d = 0$ so that $d = -(a + b + c)$. It follows that

$$\begin{aligned} bc - ad &= bc + a(a + b + c) = (a + b)(a + c), \\ ac - bd &= ac + b(a + b + c) = (a + b)(b + c), \\ ab - cd &= ab + c(a + b + c) = (a + c)(b + c). \end{aligned}$$

Hence, $|(bc - ad)(ac - bd)(ab - cd)| = (a + b)^2(a + c)^2(b + c)^2$.

Therefore the product of the three quantities $|bc-ad|, |ac-bd|, |ab-cd|$ is the square of an integer. However the product of three primes cannot be the square of an integer, so the answer to the question is "NO".

2.2. $\binom{n-i+1}{i}$ is equal to the number of i-subsets of the set $S=\{1,2,\ldots,n\}$ containing no consecutive integers. Hence the required sum is just the number a_n of subsets of S containing no consecutive integers. It can be shown easily that a_n satisfies the recurrence relation: $a_n = a_{n-1}+a_{n-2}$ with $a_0=1$ and $a_1=2$. This can also be derived from the identity:

$$\binom{n-i+1}{i} = \binom{(n-1)-i+1}{i} + \binom{(n-2)-(i-1)+1}{i-1}.$$

From this, we obtain

$$\sum_{i=0}^{\lfloor\frac{n+1}{2}\rfloor} \binom{n-i+1}{i} = \frac{5+3\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-3\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

2.3. We shall prove by induction on k that

$$\frac{n+1}{2n-k+2} < a_k < \frac{n}{2n-k} \quad \text{for} \quad k=1,2,\ldots,n.$$

For $k=1$, we have

$$a_1 = a_0 + \frac{1}{n}a_0^2 = \frac{2n+1}{4n},$$

Hence

$$\frac{n+1}{2n+1} < a_1 < \frac{n}{2n-1},$$

so the induction hypothesis is true for $k=1$.
Now suppose the induction hypothesis is true for $k=r<n$, then

$$a_{r+1} = a_r + \frac{1}{n}a_r^2 = a_r\left(1+\frac{1}{n}a_r\right).$$

Hence we have

$$\begin{aligned} a_{r+1} &> \frac{n+1}{2n-r+2}\left(1+\frac{1}{n}\cdot\frac{n+1}{2n-r+2}\right) \\ &> \frac{n+1}{2n-r+1} = \frac{n+1}{2n-(r+1)+2}. \end{aligned}$$

On the other hand,

$$a_{r+1} < \frac{n}{2n-r}\left(1+\frac{1}{n}\cdot\frac{n}{2n-r}\right) = \frac{n(2n-r+1)}{(2n-r)^2} < \frac{n}{2n-(r+1)},$$

since $(2n-r)^2 > (2n-r+1)(2n-(r+1))$. Hence the induction hypothesis is true for $k=r+1$. This completes the induction step.
When $k=n$, we get

$$1-\frac{1}{n} < 1-\frac{1}{n+2} = \frac{n+1}{n+2} < a_n < \frac{n}{2n-n} = 1,$$

the required inequality.

1997/98

1.1. Let $AC = a$, $CE = b$, $AE = c$. Applying the Ptolemy's Theorem[1] for the quadrilateral $ACEF$ we get

$$AC \cdot EF + CE \cdot AF \geq AE \cdot CF.$$

Since $EF = AF$, it implies that $\frac{FA}{FC} \geq \frac{c}{a+b}$. Similarly, $\frac{DE}{DA} \geq \frac{b}{c+a}$ and $\frac{BC}{BE} \geq \frac{a}{b+c}$. It follows that

$$\frac{BC}{BE} + \frac{DE}{DA} + \frac{FA}{FC} \geq \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{3}{2}. \qquad (1)$$

The last inequality is well known[2]. For equality to occur, we need equality to occur at every step of (1) and we need an equality each time Ptolemy's Theorem is used. The latter happens when the quadrilateral $ACEF$, $ABCE$, $ACDE$ are cyclic, that is, $ABCDEF$ is a cyclic hexagon. Also for the equality in (1) to occur, we need $a = b = c$. Hence equality occurs if and only if the hexagon is regular.

1.2. We will prove the statement by induction on n. It obviously holds for $n = 2$. Assume that $n > 2$ and that the statement is true for any integer less than n. We distinguish two cases.
Case **1**. There are no i and j such that $A_i \cup A_j = S$ and $|A_i \cap A_j| = 1$.
Let x be an arbitrary element in S. The number of sets A_i not containing x is at most $2^{n-2} - 1$ by the induction hypothesis. The number of subsets of S containing x is 2^{n-1}. At most half of these appear as a set A_i, since if $x \in A_i$, then there is no j such that $A_j = (S - A_i) \cup \{x\}$ for otherwise $|A_i \cap A_j| = 1$. Thus the number of sets A_i is at most $2^{n-2} - 1 + 2^{n-2} = 2^{n-1} - 1$.
Case **2**. There is an element $x \in S$ such that $A_1 \cup A_2 = S$ and $A_1 \cap A_2 = \{x\}$.
Let $|A_1| = r + 1$ and $|A_2| = s + 1$. Then $r + s = n - 1$. The number of sets A_i such that $A_i \subseteq A_1$ is at most $2^r - 1$ by the induction hypothesis. Similarly the number of sets A_i such that $A_i \subseteq A_2$ is at most $2^s - 1$.
If A_i is not a subset of A_1 and A_2, then $A_1 \cap A_i \neq \emptyset$, $A_2 \cap A_i \neq \emptyset$. Since $A_1 \cap A_2 \neq \emptyset$, we have $A_1 \cap A_2 \cap A_i \neq \emptyset$. Thus $A_1 \cap A_2 \cap A_i = \{x\}$. Thus $A_i = \{x\} \cup (A_i - A_1) \cup (A_i - A_2)$, and since the nonempty sets $A_i - A_1$ and $A_i - A_2$ can be chosen in $2^s - 1$ and $2^r - 1$ ways, respectively, the number of these sets is at most $(2^s - 1)(2^r - 1)$. Adding up these partial results we obtain the result that the number of A_i's is at most $2^{n-1} - 1$.

1.3. **1st solution**
Note that for any a and b, we have $(a - b)| \pm (F(a) - F(b))$. Thus 1998 divides $F(1998) - F(0)$, whence $F(1998) = F(0)$ as $|F(1998) - F(0)| \leq 1997$. Also we have $4 = 1998 - 1994$ divides $F(1994) - F(1998) = F(1994) - F(0)$, and $1994|(F(1994) - F(0))$. Thus $\text{LCM}(4, 1994) = 3988$ divides $F(1994) - F(0)$ which implies $F(1994) = F(0)$. By reversing the role of 4 and 1998, we have $F(4) = F(0)$. By considering 5 and 1993, we also have $F(1993) = F(5) = F(0)$. Then for any a, $1 \leq a \leq 1997$, we have $(x - a)|F(0) - F(a)$ for $x = 4, 5, 1993, 1994$. The least common multiplier of the 4 numbers $x - a$ is larger than 1998. Thus $F(a) = F(0)$.

2nd solution
We shall prove that the statement holds for any integer $k \geq 4$, not just $k = 1998$. Consider any polynomial $F(x)$ with integer coefficients satisfying the given inequality $0 \leq F(c) \leq k$ for every $c \in \{0, 1, \ldots, k + 1\}$. Note that $F(k + 1) = F(0)$ because $F(k + 1) - F(0)$ is a multiple of $k + 1$ not exceeding k in absolute value. Hence

$$F(x) - F(0) = x(x - k - 1)G(x),$$

1995/96

1.1. Let P be a point on the side AB of a square $ABCD$ and Q a point on the side BC. Let H be the foot of the perpendicular from B to PC. Suppose that $BP = BQ$. Prove that QH is perpendicular to HD.

1.2. For each positive integer k, prove that there is a perfect square of the form $n2^k - 7$, where n is a positive integer.

1.3. Let $S = \{0, 1, 2, \ldots, 1994\}$. Let a and b be two positive numbers in S which are relatively prime. Prove that the elements of S can be arranged into a sequence $s_1, s_2, s_3, \ldots, s_{1995}$ such that $s_{i+1} - s_i \equiv \pm a \text{ or } \pm b \pmod{1995}$ for $i = 1, 2, \ldots, 1994$.

2.1. Let C, B, E be three points on a straight line l in that order. Suppose that A and D are two points on the same side of l such that

(i) $\angle ACE = \angle CDE = 90^o$ and

(ii) CA = CB = CD.

Let F be the point of intersection of the segment AB and the circumcircle of $\triangle ADC$. Prove that F is the incentre of $\triangle CDE$.

2.2. Prove that there is a function f from the set of all natural numbers to itself such that for any natural number n, $f(f(n)) = n^2$.

2.3. Let S be a sequence $n_1, n_2, \ldots, n_{1995}$ of positive integers such that $n_1 + \cdots + n_{1995} = m < 3990$. Prove that for each integer q with $1 \le q \le m$, there is a sequence $n_{i_1}, n_{i_2}, \ldots, n_{i_k}$, where $1 \le i_1 < i_2 < \cdots < i_k \le 1995$, $n_{i_1} + \cdots + n_{i_k} = q$ and k depends on q.

1996/97

1.1. Let ABC be a triangle and let D, E and F be the midpoints of the sides AB, BC and CA respectively. Suppose that the angle bisector of $\angle BDC$ meets BC at the point M and the angle bisector of $\angle ADC$ meets AC at the point N. Let MN and CD intersect at O and let the line EO meet AC at P and the line FO meet BC at Q. Prove that $CD = PQ$.

1.2. Let a_n be the number of n-digit integers formed by 1, 2 and 3 which do not contain any consecutive 1's. Prove that a_n is equal to $(\frac{1}{2} + \frac{1}{\sqrt{3}})(\sqrt{3} + 1)^n$ rounded off to the nearest integer.

1.3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function from the set $\mathbb{R}$ of real numbers to itself. Find all such functions f satisfying the two properties:

(a) $f(x + f(y)) = y + f(x)$ for all $x, y \in \mathbb{R}$,

(b) the set $\left\{ \dfrac{f(x)}{x} \ : \ x \text{ is a nonzero real number} \right\}$ is finite.

2.1. Four integers a_0, b_0, c_0, d_0 are written on a circle in the clockwise direction. In the first step, we replace a_0, b_0, c_0, d_0 by a_1, b_1, c_1, d_1, where $a_1 = a_0 - b_0, b_1 = b_0 - c_0, c_1 = c_0 - d_0, d_1 = d_0 - a_0$. In the second step, we replace a_1, b_1, c_1, d_1 by a_2, b_2, c_2, d_2, where $a_2 = a_1 - b_1, b_2 = b_1 - c_1, c_2 = c_1 - d_1, d_2 = d_1 - a_1$. In general, at the kth step, we have numbers a_k, b_k, c_k, d_k on the circle where $a_k = a_{k-1} - b_{k-1}, b_k = b_{k-1} - c_{k-1}, c_k = c_{k-1} - d_{k-1}, d_k = d_{k-1} - a_{k-1}$. After 1997 such replacements, we set $a = a_{1997}, b = b_{1997}, c = c_{1997}, d = d_{1997}$. Is it possible that all the numbers $|bc - ad|, |ac - bd|, |ab - cd|$ are primes ? Justify your answer.

Then the circumcentre A''' of AIA_1 is the intersection of $B''C''$ with $B'C'$. Likewise the circumcentre B''' of BIB_1 is the intersection of $A''C''$ with $A'C'$ and the circumcentre C''' of CIC_1 is the intersection of $A''B''$ with $A'B'$.
First we note that the circumcentre of AIB lies on the line CI. To prove this, let the circumcircle of AIB meet CI at another point X. Then $\angle XAB = \angle XIB = \frac{1}{2}(\angle B + \angle C)$. Thus $\angle XAI = \angle XAB + \angle BAI = 90°$. Thus XI is a diameter and the circumcentre which is C'' is on the line CI. Similarly, A'' is on AI and B'' is on BI.
Now we consider the triangles $A'B'C'$ and $A''B''C''$. The lines $A'A''$, $B'B''$, and $C'C''$ are concurrent (at I), thus by Desargues' Theorem[4], the three points, namely, the intersections of $B''C''$ with $B'C'$, $A''C''$ with $A'C'$ and $A''B''$ with $A'B'$ are collinear.

3rd solution (By inversion) Let c be the incircle of $\triangle ABC$ of radius r. The image of a point X under the inversion about c is the point X^* such that $IX{\cdot}IX^* = r^2$. Inversion about a circle c has the following properties:

(a) If X lies on c, then $X^* = X$.
(b) $I^* = \infty$.
(c) If s is a circle intersecting c at two points P, Q and s passes through I, then s^* is a straight line passing through P and Q.

Now $A^* = A_o$, where A_o is the midpoint of B_1C_1. Also, ${A_1}^* = A_1$ and $I^* = \infty$. Hence, the inversion of the circumcircle of $\triangle AIA_1$ is the line A_1A_o. Similarly, the inversion of the circumcircle of $\triangle BIB_1$ is the line B_1B_o and the inversion of the circumcircle of $\triangle CIC_1$ is the line C_1C_o, where B_o is the midpoint of C_1A_1 and C_o is the midpoint of A_1B_1. Note that the 3 medians A_1A_o, B_1B_o, C_1C_o of $\triangle A_1B_1C_1$ are concurrent. Furthermore, they meet at ∞. This means that the circumcircles under consideration pass through two points. (one of them is I.) Thus they are coaxial and hence their centres are collinear.

2.2. **1st solution**
We need to prove that

$$\sqrt{\sum_{k=1}^{n} a_k} \leq \sum_{k=1}^{n-1} \sqrt{k}(\sqrt{a_k} - \sqrt{a_{k+1}}) + \sqrt{na_n}.$$

We prove this by induction on n. For $n = 1$ the void sum has value zero and the result is clear. Assume that the result holds for a certain $n \geq 1$. Consider $a_1 \geq \cdots \geq a_{n+1} \geq a_{n+2} = 0$. Write $S = \sum_{k=1}^{n} a_k$ and $b = a_{n+1}$. It suffices to prove that

$$\sqrt{S+b} - \sqrt{S} \leq -\sqrt{nb} + \sqrt{(n+1)b}.$$

This holds trivially when $b = 0$. And if $b > 0$, division by $\sqrt{b}$ takes it into the form

$$\sqrt{U+1} - \sqrt{U} \leq \sqrt{n+1} - \sqrt{n},$$

where $U = S/b$; equivalently:

$$\frac{1}{\sqrt{U+1}+\sqrt{U}} \leq \frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

Since $b = a_{n+1} \leq S/n$, we have $U \geq n$, whence the last inequality is true and the proof is complete.

2nd solution

Set $x_k = \sqrt{a_k} - \sqrt{a_{k+1}}$, for $k = 1, \dots, n$. Then

$$a_1 = (x_1 + \cdots + x_n)^2, \quad a_2 = (x_2 + \cdots + x_n)^2, \dots, a_n = x_n^2.$$

Expanding the squares we obtain

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} k x_k^2 + 2 \sum_{1 \le k < \ell \le n} k x_k x_\ell. \tag{3}$$

Note that the coefficient of $x_k x_\ell$ (where $k < \ell$) in the last sum is equal to k. The square of the right-hand side of the asserted inequality is equal to

$$\Big(\sum_{k=1}^{n} \sqrt{k} x_k\Big)^2 = \sum_{k=1}^{n} k x_k^2 + 2 \sum_{1 \le k < l \le n} \sqrt{k\ell}\, x_k x_\ell. \tag{4}$$

And since the value of (3) is obviously not greater than the value of (4), the result follows.

3rd solution

Let $c_k = \sqrt{k} - \sqrt{k-1}$, then the inequality can be transformed to

$$\sqrt{\sum_{k=1}^{n} a_k} \le \sum_{k=1}^{n} \sqrt{a_k} c_k.$$

By squaring both sides, this is in turn equivalent to

$$\sum_{k=2}^{n} a_k (c_k^2 - 1) + \sum_{0 \le i < j \le n} 2\sqrt{a_i a_j} c_i c_j \ge 0.$$

Note that $c_i c_j = \sqrt{ij} - \sqrt{i(j-1)} - \sqrt{(i-1)j} + \sqrt{(i-1)(j-1)}$. Thus for $k = 3, \dots, n$,

$$\begin{aligned}
\sum_{i=1}^{k-1} 2\sqrt{a_i a_k} c_i c_k &= \sum_{i=1}^{k-2} 2\big(\sqrt{ik} - \sqrt{i(k-1)}\big)\big(\sqrt{a_i a_k} - \sqrt{a_{i+1} a_k}\big) \\
&\quad + 2\sqrt{a_{k-1} a_k}\big(\sqrt{k(k-1)} - (k-1)\big) \\
&\ge 2\sqrt{a_{k-1} a_k}\big(\sqrt{k(k-1)} - (k-1)\big) = \sqrt{a_{k-1} a_k}\big(1 - c_k^2\big).
\end{aligned}$$

Also $2\sqrt{a_1 a_2} c_1 c_2 = \sqrt{a_1 a_2}(1 - c_2^2)$. Hence

$$\begin{aligned}
\sum_{k=2}^{n} a_k (c_k^2 - 1) &+ \sum_{0 \le i < j \le n} 2\sqrt{a_i a_j} c_i c_j \\
&\ge \sum_{k=2}^{n} a_k \big(c_k^2 - 1\big) + \sum_{k=2}^{n} \sqrt{a_{k-1} a_k}\big(1 - c_k^2\big) \\
&= \sum_{k=2}^{n} \big(1 - c_k^2\big)\big(\sqrt{a_{k-1} a_k} - a_k\big) \ge 0,
\end{aligned}$$

since $\sqrt{a_{k-1} a_k} - a_k \ge 0$ and $1 - c_k^2 \ge 0$. This completes the proof.

From solutions 2 and 3, we can conclude that equality holds if and only if there exists an index m such that $a_1 = \cdots = a_m$ and $a_k = 0$ for $k > m$.

2.3. 1st solution

We prove by induction on h, the common difference of the progression. If $h = 1$, there is nothing to prove. Fix $h > 1$ and assume that the statement is true for progressions whose common difference is less than h. Consider an arithmetic progression with first term a, and common difference h such that both x^2 and y^3 are terms in the progression. Let $d = \gcd(a, h)$. Write $h = de$. If an integer n satisfies $n \equiv a \pmod h$ and $n \geq a$, then n is a term in the progression. Thus it suffices to prove that there is a z satisfying $z^6 \equiv a \pmod h$ as this implies $(z + kh)^6 \equiv a \pmod h$ for any positive integer k and one can always choose a large k so that $(z + kh)^6 \geq a$.

Case 1. $\gcd(d, e) = 1$: We have $x^2 \equiv a \equiv y^3 \pmod h$, hence also $\pmod e$. The number e is coprime to a, hence to x and y as well. So there exists an integer t such that $ty \equiv x \pmod e$. Consequently $(ty)^6 \equiv x^6 \pmod e$, which can be rewritten as $t^6a^2 \equiv a^3 \pmod e$. Dividing by a^2 (which is legitimate because $\gcd(a, e) = 1$), we obtain $t^6 \equiv a \pmod e$. As $\gcd(d, e) = 1$, it follows that $t + ke \equiv 0 \pmod d$ for some integer k. Thus

$$(t + ke)^6 \equiv 0 \equiv a \pmod d.$$

Since $t^6 \equiv a \pmod e$, we get from the Binomial Formula

$$(t + ke)^6 \equiv a \pmod e.$$

And since d and e are coprime and $h = de$, the latter two equations imply

$$(t + ke)^6 \equiv a \pmod h.$$

Case 2. $\gcd(d, e) > 1$. Let p be a prime divisor of d and e. Assume that p^α is the greatest power of p dividing a and p^β is the greatest power of p dividing h. Recalling that $h = de$ with e being coprime to a, we see that $\beta > \alpha \geq 1$. If follows that for each term of the progression $(a + ih : i = 0, 1, \ldots)$, the greatest power of p which divides it is p^α. Since x^2 and y^3 are in the progression, α must be divisible by 2 and 3. So $\alpha = 6\gamma$ for some integer γ; hence $\alpha \geq 6$.

The progression $(p^{-6}(a + ih) : i = 1, 2, \ldots)$ with common difference $h/p^6 < h$ has integer terms and contains the numbers $(x/p^3)^2$ and $(y/p^2)^3$. By the induction hypothesis it contains a term z^6 for some integer z. Thus $(pz)^6$ is a term in the original progression. This completes the induction.

2nd solution

We use the same notation as in the first solution.

The assertion is proved by induction on h. The case $d = 1$ is trivially true.

(a) $\gcd(a, h) = 1$. (a^{-1} exists mod h.) In this case, we have $(y/x)^6 \equiv a \pmod h$.

(b) $\gcd(a, h) = r > 1$. Pick a prime p dividing r and let α be the largest positive integer such that p^α divides r. If $\alpha \geq 6$, then

$$(\frac{x}{p^2})^3 \equiv \frac{a}{p^6}, \quad (\frac{y}{p^3})^2 \equiv \frac{a}{p^6} \qquad (\text{mod } \frac{d}{p^6}).$$

By induction hypothesis, there exists z such that $z^6 \equiv \frac{a}{p^6} \quad (\text{mod } \frac{d}{p^6})$. Then $(zp)^6 \equiv a \pmod h$.

So we suppose $0 < \alpha < 6$. From $x^3 \equiv a,\ y^2 \equiv a \pmod h$, we have

$$\frac{x^3}{p^\alpha} \equiv \frac{a}{p^\alpha}, \quad \frac{y^2}{p^\alpha} \equiv \frac{a}{p^\alpha} \qquad (\text{mod } \frac{d}{p^\alpha}). \qquad (*)$$

(i) $\gcd(p, \frac{h}{p^\alpha}) = 1$. ($p^{-1}$ exists mod $\frac{d}{p^\alpha}$.) Multiply both sides of (*) by $p^{\alpha-6}$. We have

$$(\frac{x}{p^2})^3 \equiv \frac{a}{p^6}, \quad (\frac{y}{p^3})^2 \equiv \frac{a}{p^6} \qquad (\text{mod } \frac{d}{p^\alpha}).$$

By induction hypothesis, there exists z such that $z^6 \equiv \dfrac{a}{p^6} \pmod{\dfrac{d}{p^\alpha}}$. Write $a = p^\alpha a'$, then there is an integer m such that

$$(pz)^6 - p^\alpha a' = m\frac{h}{p^\alpha}.$$

Since $\alpha < 6$, p^α divides the left hand side of the equation. Thus it also divides m, whence $(pz)^6 \equiv p^\alpha a' = a \pmod{h}$.

(ii) $\gcd(p, \dfrac{h}{p^\alpha}) = p$. Then p^α is the largest power of p dividing a. Furthermore, α is a multiple of 3. To see this write $x = p^\beta x'$, where p does not divide x' and let $x = a + kh$ for some positive integer k. Then $p^{3\beta}{x'}^3 = x^3 = a + kh = p^\alpha(a' + pkh')$ for some integer a', h' with $\gcd(a', p) = 1$. Consequently, $\alpha = 3\beta$. Similarly, α is a multiple of 2. Therefore, $\alpha \geq 6$, and this case does not arise.

Footnotes

1. **Ptolemy's Theorem.** For any quadrilateral $ABCD$, we have

$$AB \cdot CD + BC \cdot DA \geq AC \cdot BD$$

and equality occurs if and only if $ABCD$ is cyclic.

2. **Proof of the inequality.** Let $x = a + b$, $y = a + c$, $z = b + c$, then

$$\frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} \geq \frac{1}{2}\left(\frac{x}{y} + \frac{y}{x} + \frac{x}{z} + \frac{z}{x} + \frac{y}{z} + \frac{z}{y} - 3\right) \geq \frac{3}{2}.$$

3. **Menelaus' Theorem.** Three points X, Y and Z on the sides BC, CA, and AB (suitably extended) of triangle ABC are collinear if and only if

$$\frac{BX}{XC} \cdot \frac{CY}{YA} \cdot \frac{AZ}{ZB} = 1.$$

4. **Desargues' Theorem.** Given any pair of triangles ABC and $A'B'C'$, the following are equivalent:

(i) The lines AA', BB' and CC' are concurrent.

(ii) The points of intersection of AB with $A'B'$, AC with $A'C'$, BC with $B'C'$ are collinear.