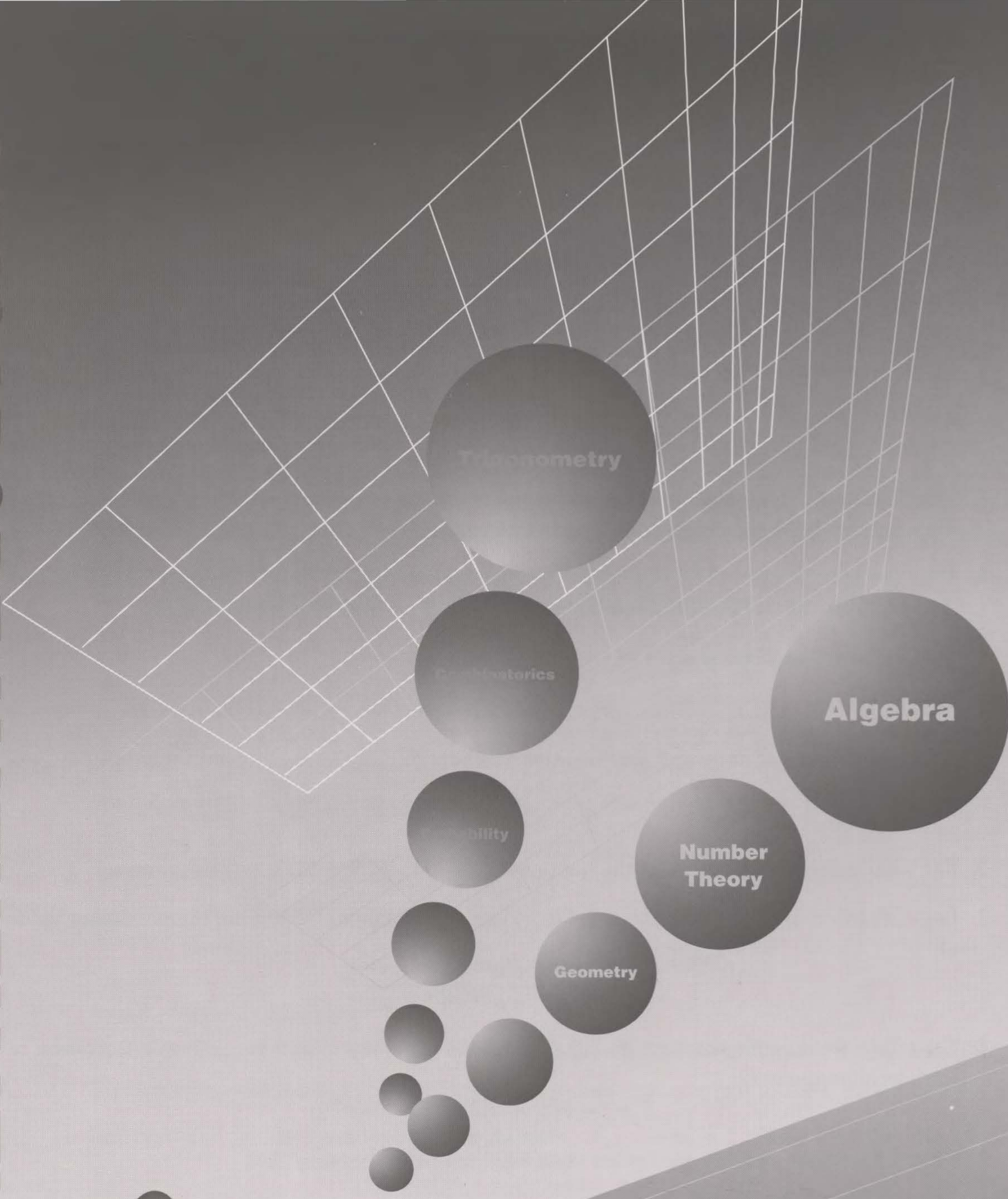
SINGAPORE MATHEMATICAL



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OLYMPIADS

Singapore Mathematical Olympiad 1994
Part A

1. Find the value x such that for any positive real number N,

$$\frac{1}{\log_2 N}+\frac{1}{\log_3 N}+\frac{1}{\log_4 N}+\cdots+\frac{1}{\log_{1994} N}=\frac{1}{\log_x N}.$$

2. Let $P(x)$ be a polynomial. Suppose the remainders of $P(x)$ when divided by x^2+2 and x^2+3 are respectively $3x+1$ and $4x+2$. Find the remainder when $P(x)$ is divided by $(x^2+2)(x^2+3)$.

3. Find the area, in cm^2, of an equilateral triangle which contains in its interior a point P whose distance from the vertices of the triangle are 3 cm, 4 cm and 5cm respectively.

4. Determine the maximum value of

$$x_1^2x_2+x_2^2x_3+\cdots+x_n^2x_1$$

for any collection of at least 3 non-negative numbers $\{x_1,x_2,\ldots,x_n\}$ such that $x_1+x_2+\cdots+x_n=1$.

5. For what integer values of N is the quotient $(N^2-71)/(7N+55)$ a positive integer?

6. Let a and b be two given numbers and x be any real number. Determine the maximum value of

$$F=\frac{(x^{2n}-a^{2n})(b^{2n}-x^{2n})}{(x^{2n}+a^{2n})(b^{2n}+x^{2n})}.$$

7. Let p be a fixed prime number. If $a(n)$ denotes the exponent of p in the prime factorization of n, determine the sum

$$S(m)=a(1)+a(2)+\cdots+a(p^m).$$

8. Find a four-digit number $abcd$ other than 8970 with the property that

$$abcd=e\times fg\times hij,$$

where e, fg, hij are numbers with one, two, and three digits respectively, and the digits a,b,c,d,e,f,g,h,i,j are all different. As an example, $8970=1\times 26\times 345$.

9. For the functions

$$\begin{aligned} f_1(x) &= \frac{2x-1}{x+1} \\ f_{n+1}(x) &= f_1(f_n(x)) \quad \text{for } n\geq 1, \end{aligned}$$

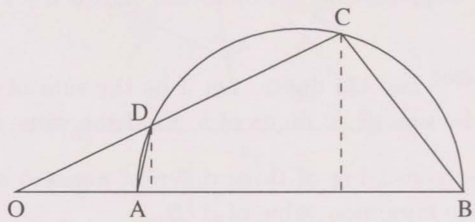
it can be shown that $f_{35}=f_5$. Find a simple expression for f_{28}.

10. The difference in the area of two similar triangles is 18 square metres, and the ratio of the larger area to the smaller is the square of an integer. The area of the smaller triangle, in square metres, is an integer, and one of its sides is 3 meteres. What is the length, in metres, of the corresponding side of the larger triangle?

Part B

1. We know that if $f(x) = (g(x))^2 + (h(x))^2$, where $g(x)$ and $h(x)$ are real polynomials, i.e., polynomials with real coefficients, then $f(x) \geq 0$ for all real numbers x. Is the converse true, that is, if $f(x)$ is a real polynomial with $f(x) \geq 0$ for all real numbers x, can we always find real polynomials $g(x)$ and $h(x)$ such that
$$f(x) = (g(x))^2 + (h(x))^2?$$
Give your arguments.

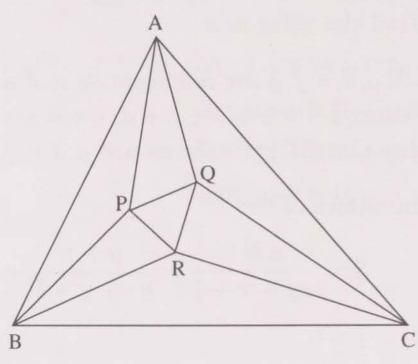
2. If O is a given point on the prolongation of diameter BA of a given semi-circle, and if ODC is a secant cutting the semi-circle at D and C, prove that the quadrilateral $ABCD$ has maximum area when the orthogonal projection of DC on AB is equal to the radius of the semi-circle.



3. We define a function f on the positive integers as follows:
$$\begin{aligned} &f(1) = 1, \quad f(3) = 3, \\ &f(2n) = f(n), \\ &f(4n+1) = 2f(2n+1) - f(n), \\ &f(4n+3) = 3f(2n+1) - 2f(n), \end{aligned}$$
for all positive integers n.
(i) For $x = a_k \times 2^k + a_{k-1} \times 2^{k-1} + \cdots + a_0$, where each a_j is either 0 or 1, evaluate $f(x)$.
(ii) Determine the number of positive integers n, less than or equal to 1994, for which $f(n) = n$.

4. Let $x_1, x_2, \ldots, x_n$ be positive numbers satisfying $x_1 x_2 \cdots x_n = 1$. Show that for any $t \geq 0$,
$$(1 + x_1 t)(1 + x_2 t) \cdots (1 + x_n t) \geq (1 + t)^n.$$

5. In $\triangle ABC$ in the following diagram, AP, AQ trisect the angle A. Likewise, BP, BR trisect $\angle B$, and CQ, CR trisect $\angle C$. Prove that $\triangle PQR$ is an equilateral triangle.



Singapore Mathematical Olympiad 1995
Part A

1. Let x, y and z be three real values such that $|x-2y|+(2y-1)^2+\sqrt{2z+4x}=0$. Find the value of $x+y+z$.

2. What are the real solutions of
$$(x+1)(x+2)(x+3)(x+4)=8?$$

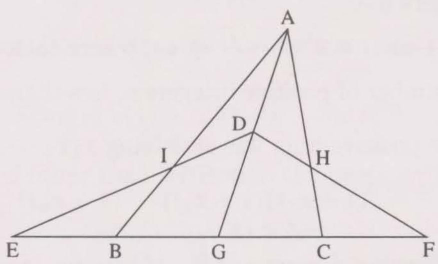
3. Let y be a perfect square consisting of four digits. Let the ones digit and the tens digit be a and b respectively. Suppose that the hundreds digit is $a+1$ and the thousands digit is b. Find the value of $\sqrt{y}$.

4. It is known that 3^{1000} has 478 digits. Let a be the sum of all digits of 3^{1000}, b the sum of all digits of a, and c the sum of all digits of b. Find the value of c.

5. Let A be a number consisting of three different nonzero digits, and B the sum of all three digits of A. Find the minimum value of A/B.

6. In a plane, three points $A(-1,0)$, $B(1,0)$, and $C(2,1)$ are given. Find the point $P(x,y)$ such that $\angle APB=45°$ and the length of PC is maximum.

7. Let D be the mid-point of the median AG of triangle ABC. GB is produced to E so that $BE=GB$ and GC is produced to F so that $CF=GC$. D is joined to E and F by straight lines which cut AB at I and CA at H. What is the ratio of the area of pentagon $DIBCH$ to that of triangle ABC?



8. Suppose that the logarithmic function $y=\log_a x$ intersects its inverse $y=a^x$ at only one point, where $a>1$. Find the value of a.

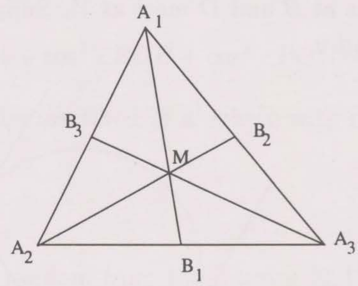
9. The seven numbers a,b,c,d,e,f,g are nonnegative real numbers that add up to 1. If M is the maximum of the five sums $a+b+c$, $b+c+d$, $c+d+e$, $d+e+f$, $e+f+g$, determine the minimum possible value that M can take as a,b,c,d,e,f,g vary.

10. Determine the extreme values of
$$S=\frac{x+1}{xy+x+1}+\frac{y+1}{yz+y+1}+\frac{z+1}{zx+z+1},$$
where $xyz=1$, and $x,y,z\geq 0$.

Part B

1. Suppose that the rational numbers a, b, and c are the roots of the equation $x^3+ax^2+bx+c=0$. Find all such rational numbers a, b and c. Justify your answer.

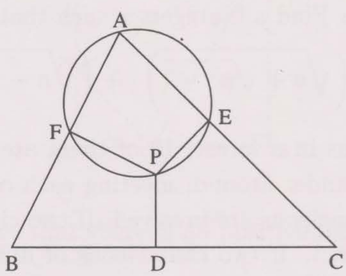
2. Let $A_1A_2A_3$ be a triangle and M an interior point. The straight lines MA_1, MA_2, MA_3 intersect the opposite sides at the points B_1, B_2, B_3 respectively. Show that if the areas of triangles A_2B_1M, A_3B_2M, and A_1B_3M are equal, then M coincides with the centroid of triangle $A_1A_2A_3$.



3. Let P be a point inside $\triangle ABC$. Let D, E, F be the feet of the perpendiculars from P to the lines BC, CA, and AB respectively. Show that

   (i) $EF = AP \sin A$,

   (ii) $PA + PB + PC \geq 2(PE + PD + PF)$.



4. Let a, b and c be positive integers such that $1 < a < b < c$. Suppose that $(ab-1)(bc-1)(ca-1)$ is divisible by abc. Find the values of a, b, and c. Justify your answer.

5. Let a, b, c, d be four positive real numbers. Prove that

$$\begin{aligned} a^{10}+b^{10}+c^{10}+d^{10} \geq\ & (0.1a+0.2b+0.3c+0.4d)^{10} \\ & +(0.4a+0.3b+0.2c+0.1d)^{10} \\ & +(0.2a+0.4b+0.1c+0.3d)^{10} \\ & +(0.3a+0.1b+0.4c+0.2d)^{10} \end{aligned}$$

Singapore Mathematical Olympiad 1996
Part A

1. There are two values of m such that the equation

$$x^4 - (3m-1)x^2 + (m-1)^2 = 0$$

has four real roots in arithmetic progression. Find the sum of those two values of m.

(A) 1 (B) $25\frac{12}{19}$ (C) $7\frac{13}{19}$ (D) $5\frac{13}{19}$ (E) $8\frac{12}{19}$

2. In Fig.1, APB, CPD are two chords of a circle that intersect at P. The tangents at A and C meet at X; the tangents at B and D meet at Y. Suppose $\angle AXC = x$ and $\angle BYD = y$. Find $\angle APD$ in terms of x and y.

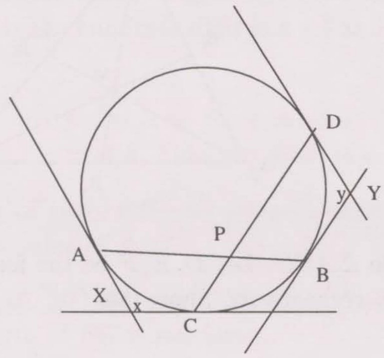


Fig. 1

3. Let $a > 1$ be an integer. Find all integers x such that

$$\left(\sqrt{a+\sqrt{a^2-1}}\right)^x + \left(\sqrt{a-\sqrt{a^2-1}}\right)^x = 2a.$$

4. There are 48 chameleons in a forest; 18 of them are white, 16 are grey and 14 are black at the beginning. They wander around, meeting each other occasionally. Suppose that at each encounter only two chameleons are involved. If two chameleons of the same colour meet, their colours remain unchanged. If two chameleons of different colours meet, both change to the third colour. At a certain instant, there are x white, y grey and z black chameleons. Which of the following cannot be equal to (x, y, z)?

(A) $(20, 15, 13)$ (B) $(28, 14, 6)$ (C) $(5, 3, 40)$ (D) $(11, 7, 30)$ (E) $(36, 10, 2)$

5. $ACEG$ is a quadrilateral circumscribed about a circle and tangent to the circle at the points B, D, F and H. See Fig. 2. If $AB = 3$, $CD = 4$, $EF = 5$ and $GH = 6$, find the radius of the circle.

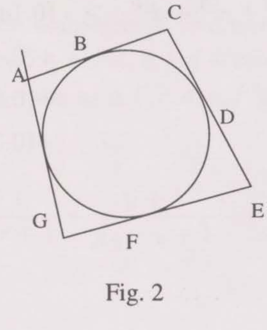


Fig. 2

6. A sequence of positive numbers $a_0, a_1, a_2, \ldots$, satisfy the relation $(a_n)^{\log a_n} = (a_{n+1})^{\log a_{n-1}}$ for $n > 1$. Find a formula for a_n if $a_0 = 2$ and $a_1 = 4$.

7. For which m is there no polyhedron with exactly m edges?

   (A) 6 (B) 7 (C) 8 (D) 9 (E) 10

8. Find the real number x such that $\sum_{k=0}^{1996} |x - \sqrt{k}|$ attains its minimum.

9. Suppose $\{x, y, z, t\} = \{60, 70, 185, 325\}$ and $xy - xz + yt = 4550$. What are x, y, z and t?

10. Determine the maximum value of

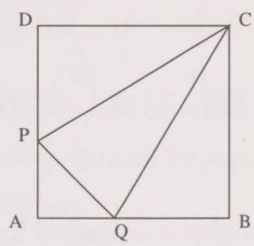
$$\cos^2 \angle POA + \cos^2 \angle POB + \cos^2 \angle POC \cos^2 \angle POD,$$

where $ABCD$ is a face of a cube inscribed in a sphere with centre O and P is any point on the sphere.

Part B

1. Three numbers are selected at random from the interval $[0, 1]$. What is the probability that they form the lengths of the sides of a triangle?

2. In the following figure, $ABCD$ is a square of unit length and P, Q are points on AD and AB respectively. Find $\angle PCQ$ if $|AP| + |AQ| + |PQ| = 2$.



3. Let n be a positive integer. Prove that there is no positive integer solution to the equation

$$(x + 2)^n - x^n = 1 + 7^n.$$

4. Determine all the solutions of the equation

$$x^3 + y^3 + z^3 = wx^2y^2z^2$$

in natural numbers x, y, z, w. Justify your answer.

Singapore Mathematical Olympaid 1997

Part A

1. Suppose r and s are the solutions of the quadratic equation $x^2+Ax+B=0$, and the equation $x^2+Cx+D=0$ has repeated solutions $r-s$. Express D in terms of A and B.

2. In Fig. 1, $ABCD$ is a quadrilateral, $\angle ABC = \angle ADC = 90°$, $\angle DAB = 60°$, $AB = 4$, $AD = 5$. Find $\dfrac{BC}{CD}$.

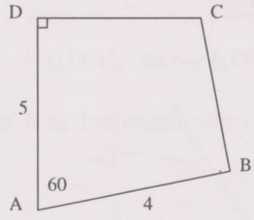


Fig. 1

3. Four numbers are chosen randomly from 1 to 100000, and the same number may be chosen more than once. Find the probability that the last digit of the product of the four numbers chosen is 1 or 9.

4. Find the area of the region in the Cartesian plane consisting of all points (x, y) such that $|x|+|y|+|x+y| \le 2$.

5. Let $f(x)$ be a polynomial of degree 5. Suppose $f(k) = 2^k$ for $k = 0, 1, 2, 3, 4, 5$. Find the value of $f(6)$.

6. The decimal part of a real number a is denoted by $\{a\}$. Find the smallest real number x such that $x > 5$ and $\{\log(x+2)\} + \{\log x\} = 1$.

7. In Fig. 2, $OABC$ is a tetrahedron. OA is perpendicular to the plane ABC. $OA = AB = AC = 2$ cm, and $BC = 2\sqrt{2}$ cm. If M and N are the mid-points of OB and OC respectively, find the area of $\triangle AMN$.

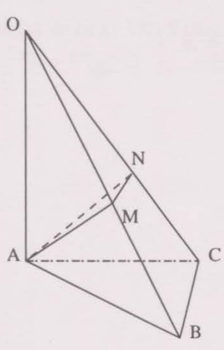


Fig. 2

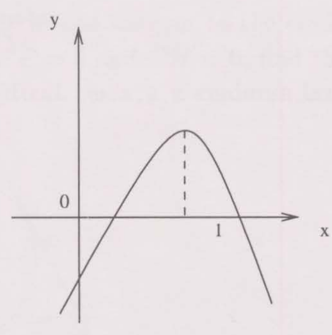


Fig. 3

8. Let a, b, c be real numbers such that the graph of $y = ax^2 + bx + c$ is as in Fig.3. How many of the following six expressions ab, ac, $a + b + c$, $a - b + c$, $2a + b$, $2a - b$ are always positive?

9. In Fig. 4, $ABCD$ is a parallelogram. AD is extended to a point F. BF meets AC and CD at E and G respectively. It is given that $BG = 60$ cm and $BF = 90$ cm. Find the length of BE.

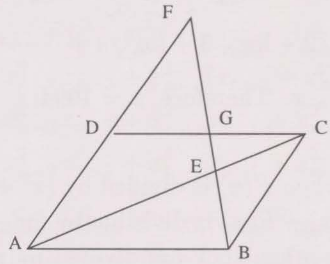


Fig. 4

10. Let $(x_1, y_1), (x_2, y_2), \cdots, (x_n, y_n)$ be all the distinct pairs of integers which satisfy the equation $(x - 5)(x - 77) = 3^y$. Find $x_1 + x_2 + \cdots + x_n$.

Part B

1. $\triangle ABC$ is an equilateral triangle. L, M and N are points on BC, CA and AB respectively. Prove that
$$MA \cdot AN + NB \cdot BL + LC \cdot CM < BC^2.$$

2. Observe that the number 4 is such that $\binom{4}{k} = \dfrac{4!}{k!(4-k)!}$ is divisible by $k + 1$ for $k = 0, 1, 2, 3$.

   Find all the natural numbers n between 50 and 90 such that $\binom{n}{k}$ is divisible by $k + 1$ for $k = 0, 1, 2, \cdots, n - 1$. Justify your answers.

3. Find all the natural numbers N which satisfy the following properties:
   (i) N has exactly 6 distinct positive factors $1, d_1, d_2, d_3, d_4, N$; and
   (ii) $1 + N = 5(d_1 + d_2 + d_3 + d_4)$.
   Justify your answers.

4. Let $n \geq 2$ be a positive integer. Suppose that $a_1, a_2, \cdots, a_n$ and $b_1, b_2, \cdots, b_n$ are $2n$ numbers such that $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1$ and
$$a_i \geq 0, \quad 0 \leq b_i \leq \frac{n-1}{n}, \quad i = 1, 2, \cdots, n.$$
   Show that
$$b_1 a_2 a_3 \cdots a_n + a_1 b_2 a_3 \cdots a_n + \cdots + a_1 a_2 \cdots a_{k-1} b_k a_{k+1} \cdots a_n$$
$$+ \cdots + a_1 a_2 \cdots a_{n-1} b_n \leq \frac{1}{n(n-1)^{n-2}}.$$