by Judith P. Jesudason

INFINITE SERIES: AN INTRODUCTION

One of the first basic facts you learn for your O-levels is the formula for the sum of a geometric progression, and its generalization, the formula for the sum of a geometric series. But if you stop and think for a minute, you'll realize that some pretty deep mathematics lies behind these and other infinite series, and the greats of years gone by had to think very carefully and critically before they came up with what we accept as fact today.

Infinite series combine the very simple notion of addition, which you learn in Primary One, with the notion of infinity, which of course, as mere humans, we will never be able to fully grasp, and can only approximate with our favorite Greek letter ε and our favorite letter for representing really large numbers, N. Lest you think only dreamy pure maths type use this subject, actually, infinite series are powerful tools used all the time in applied maths, physics, and engineering.

DEFINITION OF INFINITE SERIES

1. An *infinite sequence* is an infinite sucession of numbers which is usually given by some rule.

EXAMPLE 1

1, 4, 9, 16, ..., n^2, ..., where n is a positive integer, is an infinite sequence. The notation "..." at the end of the sequence means that the sequence continues ad infinitum, i.e. without end.

2. If $a_1, a_2, a_3, ..., a_n, ...$ is an infinite sequence, the associated *infinite series* is

$$a_1 + a_2 + a_3 + ... + a_n + ...,$$

or, to save writing,

$$\sum_{n=1}^{\infty} a_n \text{ or simply } \sum a_n.$$

Don't let the "∞" and "Σ" scare you. They are just the mathematical notations used to denote infinite series.

Note that Σ is the Greek letter for "upper-case S" which is the first letter of "Sum".

EXAMPLE 1 (revisited)

$1 + 4 + 9 + 16 + ... + n^2 + ...$ is the infinite series corresponding to our first example.

EXAMPLE 2

Let a and r be fixed real numbers. Then $a + ar + ar^2 + ar^3 + ... + ar^n + ...$ is called the *geometric series* with ratio r and constant multiple a.

HOW TO SUM AN INFINITE SERIES

How do we compute the sum of our infinite series, or, more precisely, how do we define such a sum in the first place? Here we see the fundamental difference between finite and infinite sums: a finite sum can be computed by doing a finite number of addition operations – for a sum of n numbers, we need to carry out $n - 1$ addition operations. But for infinite sums, what can we do? Clearly, we cannot keep adding numbers with a calculator forever.

Let us consider Example 1 again. It can be shown in several different ways that for a fixed positive integer n,

$$1 + 4 + 9 + ... + n^2 = \frac{n(n+1)(2n+1)}{6}$$

If you have not seen this formula, try to prove it at home. (Hint: the method of mathematical induction gives one proof).

In today's mathematics, we define the "sum" of an infinite series by a "limiting" process:

Given the infinite series

$$\sum a_n,$$

we let s_n denote the sum of the first n terms of the series,

$$s_n = a_1 + a_2 + a_3 + ... + a_n.$$

Now, if we consider the NEW *sequence of partial sums*,

$$s_1, s_2, s_3, ..., s_n, ...$$

we can try to see if the elements in this new sequence get closer and closer to some fixed (and finite!) real number L as we let the positive integer n get larger and larger. The real number L to which the sequence of partial sums tends, if it exists, is called the *limit* of the sequence.

DEFINITION

We say the series $\sum a_n$ *converges* to the sum L if the sequence of partial sums $\{s_n\}$ tends to a finite limit L, i.e., if we can get s_n as close as we want to L if we take n large enough. Otherwise, we say that the series $\sum a_n$ *diverges.*

Going back to Example 1, we see that the series

$$1 + 4 + 9 + 16 + ...$$

defined there must diverge. Why? Well, we've seen that the partial sum s_n for this series is $n(n+1)(2n+1)/6$ so that this particular s_n tends to infinity, i.e. increases without bound, as n gets larger and larger.

THE GEOMETRIC SERIES

Let's consider the geometric series $a + ar + ar^2 + ... + ar^n + ...$, where a and r are fixed real numbers, with r not equal to 1.

Recall from your O-levels the formula for an geometric progression:

If a and r are fixed real numbers, r not equal to 1,

$$a + ar + ar^2 + ... + ar^n = a\frac{(1 - r^{n+1})}{(1 - r)}$$

Therefore, the partial sum s_n for this series is given by the formula

$$s_n = a\frac{(1 - r^n)}{(1 - r)}$$

for r not equal to 1.

Now whether or not this geometric series converges depends on the size of r. In particular, by considering how r^{n+1} changes as n increases, we can prove the following theorem:

THEOREM
The geometric series $a + ar + ar^2 + \ldots$ with a not equal to 0 converges to the value $a/(1 - r)$ for $|r| < 1$, and diverges for $|r| \geq 1$.

For example, taking the following geometric series (where $a = 1$ and $r = 1/2$):

$$1 + 1/2 + 1/4 + \ldots + 1/(2)^n + \ldots,$$

our theorem above shows us that this series converges to the sum $1/(1 - 1/2) = 2$

And if we change this example and consider the case where $a = 1$ and $r = -2$, the theorem tells us that the associated series must diverge.

We have yet to discuss the cases where $r = 1$ or $r = -1$. If $r = 1$, then our geometric series becomes just

$$a + a + a + \ldots + a + \ldots,$$

so that the associated partial sum s_n for this series is given by

$$s_n = na.$$

Therefore if a is not zero, this series must diverge.

The case $r = -1$, on the other hand, plays an interesting role in the history of mathematics. Consider the geometric series with $a = 1$ and $r = -1$. The formulas for the partial sums of this series established by using the formula discussed above for geometric progressions gives

$$s_1 = 1,\ s_2 = 0,\ s_3 = 1,\ s_4 = 0,\ \ldots,$$

That is, the partial sums for this series alternate back and forth between 1 and 0 do not approach any fixed limit, hence the series diverges.

On the other hand, if we simply take the formula for the sum of a geometric series given in the above theorem, and plug in the value $a = 1$ we get

$$1/(1 - r) = 1 + r + r^2 + r^3 + \ldots .$$

Now substitute $r = -1$, to get:

$$1/(1 - (-1)) = 1/2 = 1 - 1 + 1 - 1 + 1 - 1 + \ldots . \quad (*)$$

Leibniz (1646–1716), the German mathematician best known as being co-inventor (with I. Newton) of the calculus, regrouped the terms in the right-hand side of (*) to obtain:

$$1/2 = (1 - 1) + (1 - 1) + (1 - 1) + \ldots = 0 + 0 + 0 + \ldots = 0. \quad (**)$$

Leibniz was very pleased with this second equation and proclaimed, "Thus God created the universe from nothing."

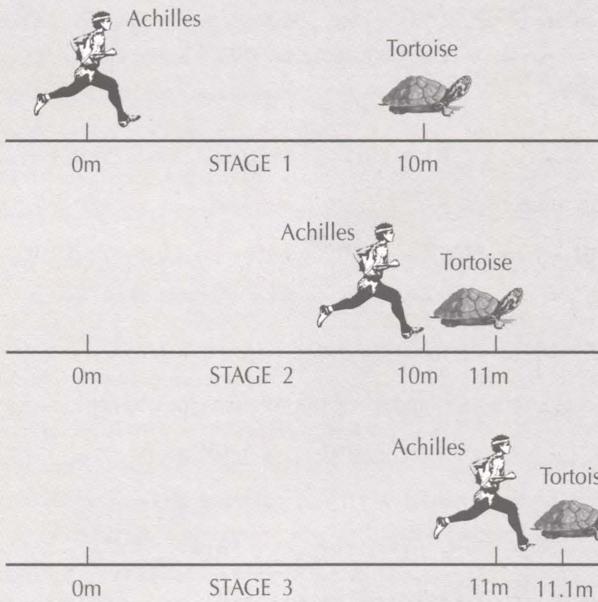
Do you have any thoughts on the problem in passing from the right hand side of (*) to the right hand size of (**)?

ZENO'S PARADOX

The ancient Greeks avoided using the notion of the infinite. One reason for this was that the Greek philosopher Zeno of Elea had put forth four paradoxes of motion which confounded thinkers for centuries! Zeno lived during the fifth century B.C. He was put to death in 430 B.C. because of his beliefs. He posed to his fellow thinkers four "paradoxes of motion". We discuss here the well-known "Achilles vs. the Tortoise" paradox and will ask you to use your critical thinking to solve this problem.

Achilles was an ancient Greek hero famed for his strength and speed. The tortoise is well-known for its slowness of motion. Of course, in any one-on-one race of Achilles vs. the tortoise, Achilles would win, even if the tortoise were allowed a head start, wouldn't you think? Well, Zeno thought NOT!

Suppose, for example, that Achilles is 10 times faster than the tortoise, and the tortoise has a 10 meter head start on Achilles.



Achilles is always getting closer and closer to the tortoise, but the tortoise is always one stage ahead. Therefore, Achilles will never catch the tortoise, even though he runs ten times faster.

Zeno argued that Achilles would never be able to catch him, by the following reasons. By the time Achilles has travelled that 10 meters, the tortoise would have travelled 1 meter, so would still be ahead of Achilles. In the time it takes Achilles to run that 1 meter the tortoise would have crawled 0.1 meter, so would still be ahead of Achilles. Using this argument, Achilles is always getting closer and closer to the tortoise, but the tortoise is always one stage ahead of Achilles. Therefore, Achilles will never catch the tortoise, ever though he is 10 times faster!

For centuries, scholars and non-scholars discussed Zeno's paradox. As late as the 19th century, some European scholars were arguing that Zeno was correct, and Achilles had lost the race!

Can you think of a resolution of this paradox that uses the notion of infinite series? Or, would you like to try to come up with a rigorous argument that shows Zeno was correct? Read on for one explanation!

RESOLUTION TO ZENO'S PARADOX USING INFINITE SERIES

Gregory of St. Vincent (1584 – 1667) was the first to use the method of infinite series to argue against Zeno's paradox, as follows:

Again, suppose Achilles is 10 times as fast as the tortoise, e.g. suppose Achilles runs 10 meters/sec. and the tortoise crawls 1 meter/sec. If the tortoise has a 10 meter head start, then at time $t = 0$, Achilles' position at $t = 0$ is $A(0) = 0$, and the tortoise's position at $t = 0$ is $T(0) = 10$m. Achilles' position at time $t = 1$ sec is $A(1) = 10$m, and the tortoise's position at time $t = 1$ is $T(1) = 11$m. Continuing on with this argument,

$$A(1 + 1/10) = 11\text{m},$$

$$T(1 + 1/10) = 11.1\text{m}.$$

At "stage n",

$$A(1 + 1/10 + 1/10^2 + ... + 1/10^n) = 10 + 1 + 1/10 + ... + 1/10^{n-1},$$

and

$$T(1 + 1/10 + 1/10^2 + ... + 1/10^n) = 10 + 1 + 1/10 + ... + 1/10^n.$$

So letting $n \to \infty$ and using our formulas for geometric series,

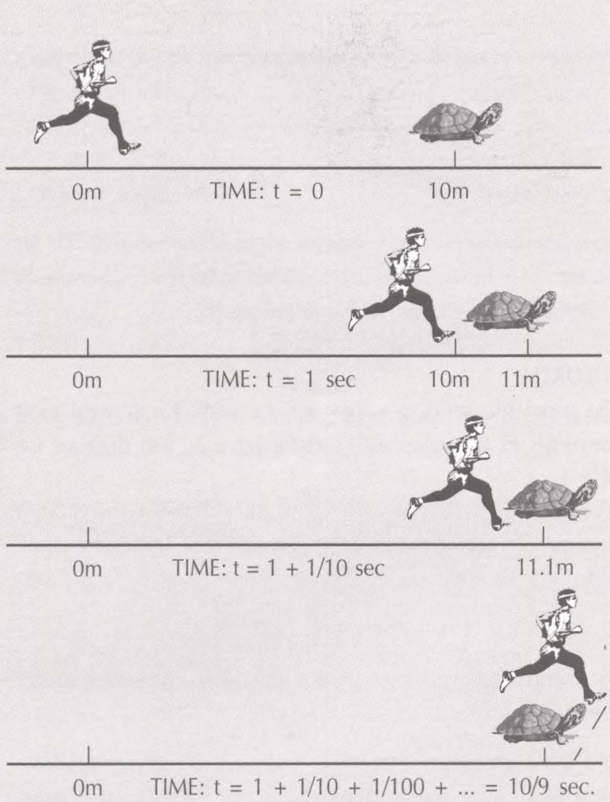
$$A(1 + 1/10 + ... + 1/10^n + ...)$$

$$= T(1 + 1/10 + ... + 1/10^n + ...)$$

$$= 10(1 + 1/10 + ... + 1/10^n + ...),$$

i.e. $$A(10/9) = T(10/9) = 100/9,$$

so that Achilles catches up with the tortoise after 10/9 seconds, at the 11 and 1/9 meter mark.



HARMONIC SERIES

From the example of the geometric series, you might guess that given a series $a_1 + a_2 + ... + a_n + ...$, if the terms a_n go to zero as $n \to \infty$ then the series will converge. On the one hand, it is true that in order for the series above to converge, it is necessary that the terms a_n go to zero as $n \to \infty$, but surprisingly enough, just the terms going to zero is not a sufficient condition for the series to converge. The most well-known example of a divergent series whose terms tend to zero is the *harmonic series*

$$1 + 1/2 + 1/3 + ... + 1/n + ...$$

or in series notation,

$$\Sigma 1/n.$$

Here, $a_n = 1/n$, and of course $1/n$ goes to zero as $n \to \infty$. But, $1 + 1/2 + ... + 1/n + ...$ diverges, since its associated sequence of partial sums increases without bound.

Oresme (1323 – 1382) was the first to note that the harmonic series diverges, by grouping the terms as follows:

Consider the partial sums $s_2 = 1 + 1/2 = 3/2$,

$$s_4 = s_{2^2} = 1 + 1/2 + (1/3 + 1/4)$$

$$\geq 1 + 1/2 + (1/4 + 1/4)$$

$$= 1 + 1/2 + 1/2 = 1 + 2/2 = 2,$$

$$s_8 = s_{2^3} = 1 + 1/2 + (1/3 + 1/4) + (1/5 + 1/6 + 1/7 + 1/8)$$

$$\geq 1 + 1/2 + (1/4 + 1/4) + (1/8 + 1/8 + 1/8 + 1/8)$$

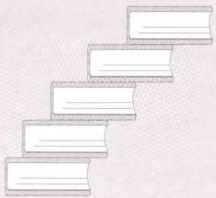
$$= 1 + 1/2 + 1/2 + 1/2 = 1 + 3/2 = 5/2.$$

In general, for any positive integer k it can be shown that $s_{2^{k+1}} \geq 1 + \frac{k+1}{2}$, which increases without bound as $n = 2^{k+1} \rightarrow \infty$.

It follows that $\sum 1/n$ diverges (but, very slowly).

STACKING BOOKS AND THE HARMONIC SERIES

Suppose you were given an arbitrarily large number of books, of uniform size and weight. If you had unlimited free time, and had a tall enough ladder, it would be possible for you to stack the books in such a way that the top book on the stack is shifted as far over to the right as you want (with respect to a fixed horizontal and vertical axis) from the horizontal position of the bottom book, without the books toppling over!



How do you do this? Well, choose the amount of overhang you want: so if the books are 1 unit long, say you want N units of overhang .

Suppose that we can get N units of overhang by stacking up n books. Now assume that you have stacked the $(n - 1)$ top books in such a way as to get maximal overhang. In order to get maximal overhang with n books, you place the already stacked $(n - 1)$ books in such a way that the center of gravity of the stack of $(n - 1)$ books is directly over the rightmost edge of the bottom book. So for example, if you are stacking 2 books, you can place the top book so that 1/2 of it hangs over the rightmost edge of the bottom book. If you are stacking three books, taking the top two books and stacking them as described above, then the center of gravity of these two books will occur 1/4 units to the left of the right corner of bottom of the two books. So, when placing the top two books on the bottom book, you can arrange things so that the top book hangs 1/2 unit over the rightmost corner of the middle book, and the middle book hangs 1/4 unit over the rightmost corner of the bottom book, for a total overhang of 1/2 + 1/4 units.

In general if you are stacking n books, you can arrange things so that the top book hangs 1/2 unit over the next book, the second book from the top hangs 1/4 unit over the third book from the top, ... the i^{th} book from the top hangs $1/(2i)$ units over the $(i + 1)^{th}$ book from the top, and ... finally, the $(n - 1)^{th}$ book hangs $1/[2(n - 1)]$ units over the n^{th} and final book on the bottom.

So, the total overhang from the right corner of the top book to the right corner of the n^{th} and bottom book is:

$$1/2 + 1/4 + 1/6 + ... + 1/[2(n - 1)] = \frac{1}{2}s_{n-1}$$

where s_{n-1} is the $(n - 1)^{th}$ partial sum of the harmonic series. Since we know the harmonic series diverges to infinity, we can get the overhang to be as large as we want!

EXERCISE

Prove that the center of gravity of the n books stacked as described above has its horizontal component occurring at $1/(2n)$ units to the left of the right-hand corner of the bottom book.

EXERCISE

How many books would you need to get a 1-unit overhang? A 2-unit overhang?

Answers: 5 books and 32 books, respectively.

MORE HISTORY ABOUT THE DEVELOPMENT OF INFINITE SERIES

It was only in the 14th century that infinite series began to be commonly used, and even then, there were no formal definitions of divergence and convergence. Mathematicians such as Isaac Newton (1642-1727), Brook Taylor (1685-1731), Leibniz, Euler (1707-1783), and others manipulated series very freely to much mathematical advantage, without worrying too much about whether they converged or diverged. They generalized the notion of infinite series of real numbers to the notion of "power series", which were expansions in terms of the monomials, for especially nice functions, of x (these days known as *analytic* functions). The series expansions for the logarithmic and trigonometric functions, for example, gave rise to greater accuracy in the calculation of specific values of these functions, which in turn helped to promote advances in navigation.

The mathematician Abel (1801-1829) clarified many aspects of the convergence and divergence of power series. He was very dubious about the use of divergent series in calculations:

"The divergent series are the invention of the devil ... by using them, one may draw any conclusion whatsoever ...".

We have already seen an example of this with Leibniz' "creation of the universe from nothing"!

It was the French mathematican A. Cauchy (1789-1857) who was the first to give a truly rigorous definition of the notions of convergence and divergence of series. He virtually "banned" the use of divergent series in proofs, and established his famous "Cauchy Criterion" for convergence of series.

Cauchy's work was revolutionary for his day and made some of his colleagues very nervous indeed! For example, it is said that after Cauchy first presented his theory of series in a colloquium, Laplace (1749-1827) went into seclusion to check all the series in his massive book "Celestial Mechanics" which had recently been published. Fortunately for him, they all satisfied Cauchy's definition of convergence.

OTHER IMPORTANT NAMES IN THE HISTORY OF SERIES

J. Fourier (1768-1830): he developed the theory of Fourier series which are widely used in physics and engineering (e.g. in signal processing).

K. Weierstrass (1815-1897): he laid the foundations of modern analysis as it is studied today. He cleared up many misconceptions by developing rigourously the notion of "uniform convergence" for series whose terms are functions (e.g., power series and Fourier series).

P. G. L. Dirichlet (1805-1859): he studied Fourier series, which led him into foundations of the real number system.

B. Riemann (1826-1866): he developed the famous Riemann zeta function, which has connections to the harmonic series.

G. Cantor (1845-1918): his study of Fourier series led him into the study of all sorts of unusual sets, including the famous "Cantor set". M^2

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Why no Nobel Prize in mathematics?

contributed by Yan Kow Cheong

A century-old mathematical rumour runs as follows:

Alfred Nobel (1833 - 1896), the founder of the Nobel Prize, didn't establish a Nobel Prize in mathematics because he wanted to retaliate against **Gustav Magnus Mittag-Leffler** (1846-1927), a likely winner at the time of inception of the Prize, who was reported to be having an affair with Nobel's wife.

No doubt the gossip about a love affair aroused the interest of mathematicians, but the catch is that Nobel never married.

Another version of this gossip claims that Mittage–Leffler, a man of considerable wealth, antagonised Nobel. The chemist, afraid that Mittag-Leffler, a leading Swedish mathematician, might win a Nobel prize in mathematics, then refused to institute such a prize.

Whatever the reason for the lack of a Nobel Prize, great mathematicians do not go unrewarded. Since 1936, the **Fields Medal**, the equivalent of a Nobel Prize in mathematics in terms of honour if not in terms of financial reward, has been awarded every four years to outstanding mathematicians below the age of 40. Candidates are chosen based on their solution of difficult problems and the creation of new theories and methods which expand the fields of application of mathematics.

Mr. Yan Kow Cheong is currently involved in the development of educational mathematics software, and he conducts recreational mathematics courses regularly.